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The Restricted Detour Polynomial of the Theta Graph

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ABSTRACT

The restricted detour distance $D^*(u,v)$ between two vertices u and v of a connected graph G is the length of a longest $u-v$ path P in G such that $\langle V(P)\rangle = P$. The main goal of this paper is to obtain the restricted detour polynomial of the theta graph. Moreover, the restricted detour index of the theta graph will also be obtained.
Keywords: Restricted detour distance, restricted detour polynomial, Theta graphs.

متعددة حدود الألتفاف المقيدة للبيان ثيتا

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الملخص

تعرف مسافة الألتفاف المقيدة $D^*(u,v)$ لرأسين u و v في البيان المتصل G على أنها الطول لأطول درب P في G بين الرأسين u و v والذي يحقق ألشرط $\langle V(P)\rangle = P$. أن الهدف الرئيسي لهذا البحث هو أيجاد متعددة حدود الألتفاف المقيد للبيان ثيتا، وكذلك تم الحصول على دليل الألتفاف المقيدة للبيان ثيتا.
الكلمات المفتاحية: مسافة الالتفاف المقيدة، متعددة حدود الالتفاف المقيدة، بيانات ثيتا.

1 Introduction

In this paper, we are concerned only with finite connected simple graphs. We refer the reader to [1,3,4,5,6] for details on graphs, distances in graphs and graph based polynomials. The idea of the restricted detour polynomials was first introduced by Abdullah and Muhammed-Salih[2]. They obtained the restricted detour polynomials and restricted detour indices of some compound graphs.

Let G be a connected graph, the (standard) **distance** between two vertices u and v of G, denoted $d(u,v)$, is the number of edges in a shortest u-v path in G. The **restricted detour distance** $D^*(u,v)$ between two vertices u and v of G is the length of a longest $u-v$ path P in G such that $\langle V(P)\rangle = P$. An induced u-v path of length $D^*(u,v)$ is called a **detour path** [4]. The **restricted detour polynomial** [2,8] of the graph G, denoted by $D^*(G;x)$ is defined as follows

$D^*(G;x) = \sum_{u,v} x^{D^*(u,v)}$,

where the summation is taken over all unordered pairs u, v of vertices of G.Moreover, one easily notice that $D^*(G;x) = \sum_{k\geq 0} C^*(G,k)x^k$, in which $C^*(G,k)$ is the number of unordered pairs of vertices u, v of G such that $D_G^*(u,v) = k$.

Let u be any vertex of G, and let $C^*(u,G;k)$be the number of vertices v of G such that $D^*(u,v) = k$. Then, the polynomial defined by

$$D^*(u,G;x) = \sum_{k\geq 0} C^*(u,G;k)x^k,$$

is called the restricted detour polynomial of vertex u.

It is clear that $D^*(G;x) = \frac{1}{2}(\sum_{u\in V(G)} D^*(u,G;x) + p)$.

Let P_k and C_k denote the path and the cycle with k vertices, respectively. The restricted detour polynomials of P_k and C_k is obtained in [2] and given in the following proposition.

Proposition 1.1

(1) $D^*(P_k;x) = \sum_{i=0}^{k-1}(k-i)x^i$.

(2) $$D^*(C_k;x) = \begin{cases} k(1 + x + \sum_{i=\frac{k+1}{2}}^{k-2} x^i) & if\ k\ is\ odd, \\ k(1 + x + \frac{1}{2}x^{k/2} + \sum_{i=\frac{k}{2}+1}^{k-2} x^i) & if\ k\ is\ even. \end{cases}$$

2 The Restricted Detour Polynomial of the Theta Graph

The **theta graph**[7] $\theta(l,m,n)$ is the graph consisting of three internally disjoint paths with common endpoints *z* and *y* and lengths $l+1$, $m-1$ and $n-1$ as depicted in Figure 2.1(a). In this paper, we focus our attention on the theta graph $\theta(0,m,n)$ or simply $\theta(m,n)$, as shown in Figure 2.1(b). Without loss of generality, we assume $m \leq n$.

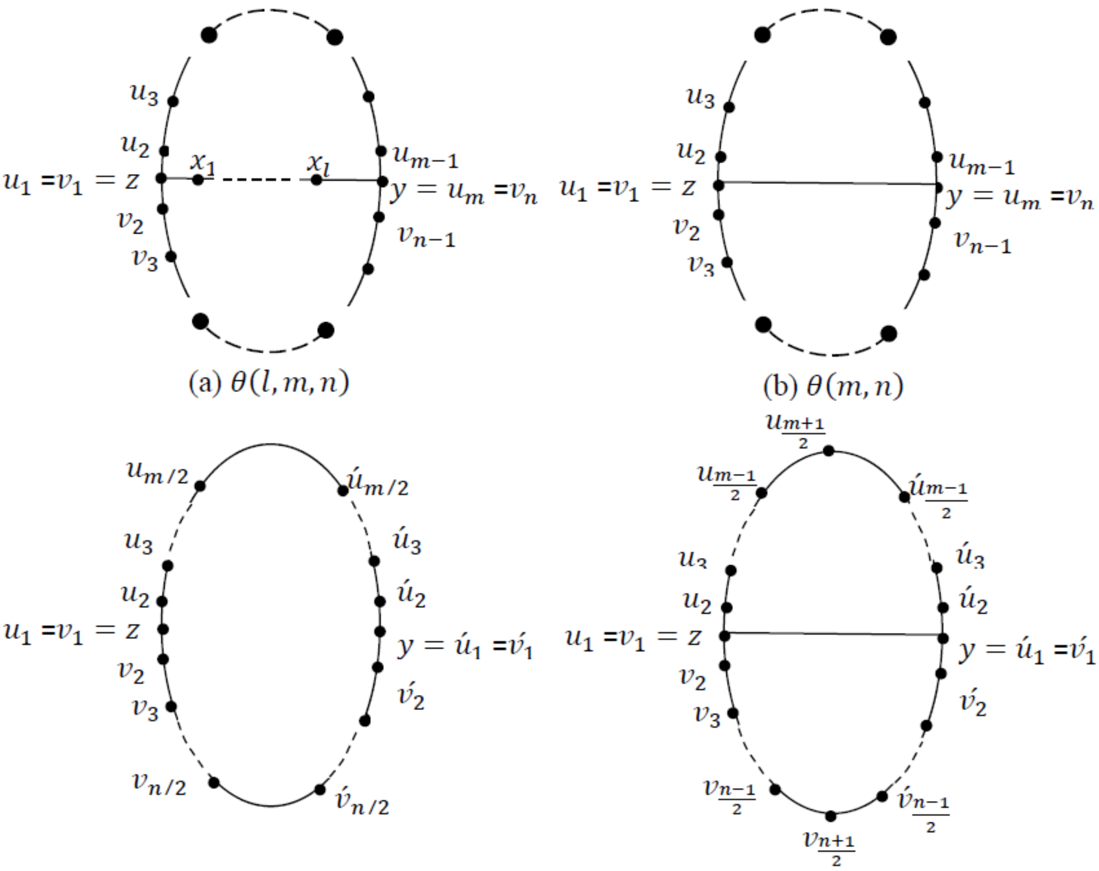

Figure 2.1: The Theta Graph

By simple calculations, we obtain

$D^*(\theta(3,3);x) = 4 + 5x + x^2$,
$D^*(\theta(3,4);x) = 5 + 6x + 2x^2 + 2x^3$,
$D^*(\theta(4,4);x) = 6 + 7x + 4x^2 + 2x^3 + 2x^4$,
$D^*(\theta(4,5);x) = 7 + 8x + 2x^2 + 5x^3 + 4x^4 + 2x^5$,
$D^*(\theta(5,5);x) = 8 + 9x + 10x^3 + 2x^4 + 5x^5 + 2x^6$, and
$D^*(\theta(5,6);x) = 9 + 10x + 8x^3 + 6x^4 + 4x^5 + 6x^6 + 2x^7$.

The restricted detour polynomials of the theta graph $\theta(m,n)$ are obtained in the next results.

Theorem 2.1 For even $m, n \geq 6$, we have

$$D^*(\theta(m,n);x) = D^*(C_m;x) + D^*(C_n;x) - x - 2 - 2x^{n-1} + 2\sum_{i=2}^{\frac{m}{2}}\sum_{j=2}^{\frac{n}{2}} x^{m+n-(i+j)} + 2\sum_{i=3}^{\frac{m}{2}}\sum_{j=3}^{\frac{n}{2}} x^{m+n+1-(i+j)} + 2\sum_{j=2}^{\frac{n-m}{2}+2} x^{n-j+1} + 2\sum_{j=\frac{n-m}{2}+3}^{\frac{n}{2}} x^{m-3+j} + 2\sum_{i=2}^{\frac{m}{2}} x^{n-3+i},$$

in which, C_p is a cycle of p vertices.

Proof. Let u and v be any two vertices of $V(\theta(m,n))$. We refer to Figure 2.1(c), and denote

$V_1 = \left\{u_2, u_3, \dots, u_{\frac{m}{2}}\right\}, \acute{V_1} = \left\{\acute{u_2}, \acute{u_3}, \dots, \acute{u}_{\frac{m}{2}}\right\}, V_2 = \left\{v_2, v_3, \dots, v_{\frac{n}{2}}\right\}$ and $\acute{V_2} = \left\{\acute{v_2}, \acute{v_3}, \dots, \acute{v}_{\frac{n}{2}}\right\}$.

Two main cases can be distinguished for *u* and *v*

Case I For all possibilities of $u, v \in V_1 \cup \acute{V_1} \cup \{u_1, \acute{u_1}\}$ (or $u, v \in V_2 \cup \acute{V_2} \cup \{v_1, \acute{v_1}\}$) and notice that the pair (z, y) with $D^*(z, y) = 1$ and each of the vertices z and y are counted twice, we have the corresponding polynomial $D^*(C_m; x) + D^*(C_n; x) - x - 2$.

Case II If $u \in V_1 \cup \acute{V_1}$ and $v \in V_2 \cup \acute{V_2}$, then there are four subcases can be distinguished for u and v

(1) If $u \in V_1$ and $v \in V_2$,then it is obvious that the path P_1,
$P_1: u = u_i, u_{i+1}, \dots, u_{\frac{m}{2}}, \acute{u}_{\frac{m}{2}}, \dots, \acute{u_2}, \acute{u_1}, \acute{v_2}, \dots, \acute{v}_{\frac{n}{2}}, v_{\frac{n}{2}}, \dots, v_j = v$ is a longest $u - v$ path with $\langle P_1 \rangle = P_1$, for $i = 2, \dots, m/2$ and $j = 2, \dots, n/2$.
Evidently, $D^*(u, v) = D^*(u_i, v_j) = m - i + n - j = m + n - (i + j)$.
Similarly, if $u \in \acute{V_1}$ and $v \in \acute{V_2}$, we have
$D^*(u, v) = D^*(\acute{u_i}, \acute{v_j}) = m + n - (i + j)$, for $i = 2, \dots, m/2$ and $j = 2, \dots, n/2$.
Now, for all values of *i* and *j*, the corresponding polynomial is

$$F_1(x) = 2 \sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n-(i+j)}.$$

(2) If $u \in V_1 - \{u_2\}$ and $v \in \acute{V_2} - \{\acute{v_2}\}$, then the path P_2
$P_2: u = u_i, u_{i+1}, \dots, u_{\frac{m}{2}}, \acute{u}_{\frac{m}{2}}, \dots, \acute{u_2}, \acute{u_1}, u_1 (= v_1), v_2, \dots, v_{\frac{n}{2}}, \acute{v}_{\frac{n}{2}}, \dots, \acute{v_j} = v$ is a longest $u - v$ path with $\langle P_2 \rangle = P_2$, for $i = 3, \dots, m/2$ and $j = 3, \dots, n/2$.
In this case, $D^*(u, v) = D^*(u_i, \acute{v_j}) = m + n - (i + j) + 1$.
Similarly, if $u \in \acute{V_1} - \{\acute{u_2}\}$ and $v \in V_2 - \{v_2\}$, then
$D^*(u, v) = D^*(\acute{u_i}, v_j) = m + n - (i + j) + 1$, for $i = 3, \dots, m/2$ and $j = 3, \dots, n/2$.
Now, for all possible values of *i* and *j*, the corresponding polynomial is

$$F_2(x) = 2 \sum_{i=3}^{\frac{m}{2}} \sum_{j=3}^{\frac{n}{2}} x^{m+n+1-(i+j)}.$$

(3) If $u = u_2$ and $v \in \acute{V_2}$; and since $m \leq n$ then there are two subcases can be distinguished

(a) For $j = 2, \dots, \frac{n-m}{2} + 2$, the path $P_3: u = u_2, u_1 (= v_1), v_2, \dots, v_{\frac{n}{2}}, \acute{v}_{\frac{n}{2}}, \dots, \acute{v_j} = v$ is a longest $u - v$ path with $\langle P_3 \rangle = P_3$ and has length $n - j + 1$. Hence, $D^*(u, v) = D^*(u_2, \acute{v_j}) = n - j + 1$.

(b) For $j = \frac{n-m}{2} + 3, \dots, \frac{n}{2}$, the path $\acute{P_3}: u = u_2, u_3, \dots, u_{\frac{m}{2}}, \acute{u}_{\frac{m}{2}}, \dots, \acute{u_2}, \acute{u_1} (= \acute{v_1}), \acute{v_2}, \dots, \acute{v_j} = v$ is a longest $u - v$ path with $\langle \acute{P_3} \rangle = \acute{P_3}$, and has length $m - 3 + j$.
Hence, $D^*(u, v) = D^*(u_2, \acute{v_j}) = m - 3 + j$.
Similarly, if $u = \acute{u_2}$ and $v \in V_2$ then

$$D^*(u, v) = D^*(\acute{u_2}, v_j) = \begin{cases} n - j + 1 \ if \ j = 2, \dots, \frac{n-m}{2} + 2, \\ m - 3 + j \ if \ j = \frac{n-m}{2} + 3, \dots, \frac{n}{2}. \end{cases}$$

Notice that, each of the pairs $(u_2, \acute{v_2})$ and $(\acute{u_2}, v_2)$ are counted twice with $D^*(u_2, \acute{v_2}) = D^*(\acute{u_2}, v_2) = n - 1$.
Now, for all possible values of *i* and *j*, the corresponding polynomial is

$F_3(x) = 2\sum_{j=2}^{\frac{n-m}{2}+2} x^{n-j+1} + 2\sum_{j=\frac{n-m}{2}+3}^{\frac{n}{2}} x^{m-3+j} - 2x^{n-1}.$

(4) If $u \in \acute{V}_1$ and $v = v_2$ (or $u \in V_1$ and $v = \acute{v}_2$), then
$D^*(u,v) = D^*(\acute{u}_i, v_2) = D^*(u_i, \acute{v}_2) = n - 2 + i - 1 = n - 3 + i,$ for $i = 2, \dots, m/2$

This produces the polynomial $F_4(x) = 2\sum_{i=2}^{\frac{m}{2}} x^{n-3+i}$.

Adding the polynomials obtained from the cases I and II and simplifying, we get the required result.■

Theorem 2.2 For odd $m, n \geq 7$, we have

$$D^*(\theta(m,n);x) = D^*(C_m;x) + D^*(C_n;x) + 2x^{\frac{n-1}{2}+m-2} - x - 2 - 2x^{n-1} + x^{\frac{m+n}{2}}$$
$$+ 2x^{\frac{m-1}{2}+n-2} + 2\sum_{i=2}^{\frac{m-1}{2}}\sum_{j=2}^{\frac{n-1}{2}} x^{m+n-(i+j)} + 2\sum_{i=3}^{\frac{m-1}{2}}\sum_{j=3}^{\frac{n-1}{2}} x^{m+n+1-(i+j)} + 2\sum_{j=2}^{\frac{n-m}{2}+2} x^{n-j+1}$$
$$+2\sum_{j=\frac{n-m}{2}+3}^{\frac{n-1}{2}} x^{m-3+j} + 2\sum_{i=2}^{\frac{m-1}{2}} x^{n-3+i} + 2\sum_{j=3}^{\frac{n-1}{2}} x^{\frac{m-1}{2}+1+n-j} + 2\sum_{i=3}^{\frac{m-1}{2}} x^{\frac{n-1}{2}+1+m-i}$$

Proof. Let u and v be any two vertices of $V(\theta(m,n))$. We refer to Figure 2.1(d), and denote
$V_1 = \left\{u_2, u_3, \dots, u_{\frac{m-1}{2}}\right\}, \acute{V}_1 = \left\{\acute{u}_2, \acute{u}_3, \dots, \acute{u}_{\frac{m-1}{2}}\right\}, V_2 = \left\{v_2, v_2, \dots, v_{\frac{n-1}{2}}\right\}$ and $\acute{V}_2 = \left\{\acute{v}_2, \acute{v}_3, \dots, \acute{v}_{\frac{n-1}{2}}\right\}$

Two main cases can be distinguished for *u* and *v*

<u>**Case I**</u> For all possibilities of $u, v \in V_1 \cup \acute{V}_1 \cup \{u_1, \acute{u}_1\}$ (or $u, v \in V_2 \cup \acute{V}_2 \cup \{v_1, \acute{v}_1\}$) and notice that the pair (z, y) with $D^*(x,y) = 1$ and each of the vertices are counted twice, we have the corresponding polynomial $D^*(C_m;x) + D^*(C_n;x) - x - 2$.

<u>**Case II**</u> If $u \in V_1 \cup \acute{V}_1$ and $v \in V_2 \cup \acute{V}_2$, then there are nine subcases can be distinguished for u and v

(1) If $u \in V_1$ and $v \in V_2$,then it is obvious that the path P_1,
$P_1: u = u_i, u_{i+1}, \dots, u_{\frac{m-1}{2}}, u_{\frac{m+1}{2}}, \acute{u}_{\frac{m-1}{2}}, \dots, \acute{u}_2, \acute{u}_1, \acute{v}_2, \dots, \acute{v}_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n-1}{2}}, \dots, v_j = v$ is a longest $u - v$ path with $\langle P_1 \rangle = P_1$, for $i = 2, \dots, (m-1)/2$ and $j = 2, \dots, (n-1)/2$.
Evidently,$D^*(u,v) = D^*(u_i, v_j) = m - i + n - j = m + n - (i + j)$.
Similarly, if $u \in \acute{V}_1$ and $v \in \acute{V}_2$, we have $D^*(u,v) = D^*(\acute{u}_i, \acute{v}_j) = m + n - (i + j)$, for $i = 2, \dots, (m-1)/2$ and $j = 2, \dots, (n-1)/2$.
Now, for all such possible values of *i* and *j*, the corresponding polynomial is
$F_1(x) = 2\sum_{i=2}^{\frac{m-1}{2}}\sum_{j=2}^{\frac{n-1}{2}} x^{m+n-(i+j)}$.

(2) If $u \in V_1 - \{u_2\}$ and $v \in \acute{V}_2 - \{\acute{v}_2\}$, then the path P_2
$P_2: u = u_i, u_{i+1}, \dots, u_{\frac{m-1}{2}}, u_{\frac{m+1}{2}}, \acute{u}_{\frac{m-1}{2}}, \dots, \acute{u}_2, \acute{u}_1, u_1(= v_1), v_2, \dots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, \acute{v}_{\frac{n-1}{2}}, \dots, \acute{v}_j = v$ is a longest $u - v$ path with $\langle P_2 \rangle = P_2$, for $i = 3, \dots, (m-1)/2$ and $j = 3, \dots, (n-1)/2$.

In this case, $D^*(u,v) = D^*(u_i, \acute{v}_j) = m + n - (i+j) + 1$.
Similarly, if $u \in \acute{V}_1 - \{\acute{u}_2\}$ and $v \in V_2 - \{v_2\}$, then
$D^*(u,v) = D^*(\acute{u}_i, v_j) = m + n - (i+j) + 1$, for $i = 3, \dots, (m-1)/2$ and $j = 3, \dots, (n-1)/2$.
Now, for all such possible values of *i* and *j*, the corresponding polynomial is
$F_2(x) = 2\sum_{i=3}^{\frac{m-1}{2}} \sum_{j=3}^{\frac{n-1}{2}} x^{m+n+1-(i+j)}$.

(3) If $= u_2$ and $v \in \acute{V}_2$; and since $m \le n$ then there are two subcases can be distinguished

(a) For $j = 2, \dots, \frac{n-m}{2} + 2$, the path
$P_3: u = u_2, u_1 = v_1, v_2, \dots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, \acute{v}_{\frac{n-1}{2}}, \dots, \acute{v}_j = v$ is a longest $u - v$ path with $\langle P_3 \rangle = P_3$ and has length $n - j + 1$.
Hence, $D^*(u,v) = D^*(u_2, \acute{v}_j) = n - j + 1$.

(b) For $j = \frac{n-m}{2} + 3, \dots, \frac{n-1}{2}$, the path
$\acute{P}_3: u = u_2, u_3, \dots, u_{\frac{m-1}{2}}, u_{\frac{m+1}{2}}, \acute{u}_{\frac{m-1}{2}}, \dots, \acute{u}_2, \acute{u}_1 (= \acute{v}_1), \acute{v}_2, \dots, \acute{v}_j = v$ is a longest $u - v$ path with $\langle \acute{P}_3 \rangle = \acute{P}_3$, and has length $m - 3 + j$. Hence, $D^*(u,v) = D^*(u_2, \acute{v}_j) = m - 3 + j$. Similarly, if $u = \acute{u}_2$ and $v \in V_2$ then

$$D^*(u,v) = D^*(\acute{u}_2, v_j) = \begin{cases} n - j + 1 \ if \ j = 2, \dots, \frac{n-m}{2} + 2, \\ m - 3 + j \ if \ j = \frac{n-m}{2} + 3, \dots, \frac{n-1}{2}. \end{cases}$$

Notice that, each of the pairs $(u_2, \acute{v}_2)$ and $(\acute{u}_2, v_2)$ are counted twice with $D^*(u_2, \acute{v}_2) = D^*(\acute{u}_2, v_2) = n - 1$.
Now, for all such possible values of *i* and *j*, the corresponding polynomial is
$F_3(x) = 2\sum_{j=2}^{\frac{n-m}{2}+2} x^{n-j+1} + 2\sum_{j=\frac{n-m}{2}+3}^{\frac{n-1}{2}} x^{m-3+j} - 2x^{n-1}$.

(4) If $u \in \acute{V}_1$ and $v = v_2$ (or $u \in V_1$ and $v = \acute{v}_2$), then
$D^*(u,v) = D^*(\acute{u}_i, v_2) = D^*(u_i, \acute{v}_2) = n - 2 + i - 1 = n - 3 + i$, for $i = 2, 3, \dots, (m-1)/2$

This produces the polynomial $F_4(x) = 2\sum_{i=2}^{\frac{m-1}{2}} x^{n-3+i}$.

(5) If $u = u_{\frac{m+1}{2}}$ and $v = v_{\frac{n+1}{2}}$, then ,
$D^*(u,v) = D^*(u_{\frac{m+1}{2}}, v_{\frac{n+1}{2}}) = \frac{m-1}{2} + 1 + \frac{n-1}{2} = \frac{m+n}{2}$,and the polynomial is
$F_5(x) = x^{\frac{m+n}{2}}$.

(6) If $u = u_{\frac{m+1}{2}}$ and $v = v_2$ (or $u = u_{\frac{m+1}{2}}$ and $v = \acute{v}_2$), then ,
$D^*(u,v) = D^*(u_{\frac{m+1}{2}}, v_2) = D^*(u_{\frac{m+1}{2}}, \acute{v}_2) = \frac{m-1}{2} + n - 2$.

This produces the polynomial $F_6(x) = 2\,x^{\frac{m-1}{2}+n-2}$.

(7) If $u = u_{\frac{m+1}{2}}$ and $v \in V_2 - \{v_2\}$ (or $u = u_{\frac{m+1}{2}}$ and $v \in \acute{V}_2 - \{\acute{v}_2\}$), then ,
$D^*(u,v) = D^*(u_{\frac{m+1}{2}}, v_j) = D^*(u_{\frac{m+1}{2}}, \acute{v}_j) = \frac{m-1}{2} + n + 1 - j$,and the polynomial is
$F_7(\mathrm{x}) = 2\sum_{j=3}^{\frac{n-1}{2}} x^{\frac{m-1}{2}+n+1-j}$.

(8) If $u = u_2$ and $v = v_{\frac{n+1}{2}}$ (or $u = \acute{u}_2$ and $v = v_{\frac{n+1}{2}}$), then

$d^*(u,v) = D^*(u_2, v_{\frac{n+1}{2}}) = \frac{n-1}{2} + m - 2$, and this gives us the polynomial

$F_8(\mathrm{x}) = 2\, x^{\frac{n-1}{2}+m-2}$.

(9) If $u \in V_1 - \{u_2\}$ and $v = v_{\frac{n+1}{2}}$ (or $u \in \acute{V}_1 - \{\acute{u}_2\}$ and $v = v_{\frac{n+1}{2}}$), then

$D^*(u,v) = D^*(u_i, v_{\frac{n+1}{2}}) = D^*(\acute{u}_i, v_{\frac{n+1}{2}}) = \frac{n-1}{2} + m + 1 - i$, and this gives us the polynomial

$F_9(\mathrm{x}) = 2\, \sum_{i=3}^{\frac{m-1}{2}} x^{\frac{n-1}{2}+m+1-i}$.

Now, adding the polynomials obtained from the Cases I and II and simplifying, we get the required result.■

Using the same procedure followed inproving Theorem 2.1 and Theorem 2.2 we obtain the following results.

Theorem 2.3 For odd $m \geq 7$ and even $n \geq 8$, we have

$$D^*(\theta(m,n);x) = D^*(C_m;x) + D^*(C_n;x) - x - 2 - 2x^{n-1} + 2\,x^{\frac{m-1}{2}+n-2} + 2\sum_{i=2}^{\frac{m-1}{2}}\sum_{j=2}^{\frac{n}{2}} x^{m+n-(i+j)} + 2\sum_{i=3}^{\frac{m-1}{2}}\sum_{j=3}^{\frac{n}{2}} x^{m+n+1-(i+j)} + 2\sum_{j=2}^{\frac{n-m-1}{2}+2} x^{n-j+1} + 2\sum_{j=\frac{n-m-1}{2}+3}^{\frac{n}{2}} x^{m-3+j} + 2\sum_{i=2}^{\frac{m-1}{2}} x^{n-3+i} + 2\sum_{j=3}^{\frac{n}{2}} x^{\frac{m-1}{2}+1+n-j}. \blacksquare$$

Theorem 2.4 For even $m \geq 6$ and odd $n \geq 7$, we have

$$D^*(\theta(m,n);x) = D^*(C_m;x) + D^*(C_n;x) - x - 2 - 2\,x^{n-1} + 2\,x^{\frac{n-1}{2}+m-2} + 2\sum_{i=2}^{\frac{m}{2}}\sum_{j=2}^{\frac{n-1}{2}} x^{m+n-(i+j)} + 2\sum_{i=3}^{\frac{m}{2}}\sum_{j=3}^{\frac{n-1}{2}} x^{m+n+1-(i+j)} + 2\sum_{j=2}^{\frac{n-1-m}{2}+2} x^{n-j+1} + 2\sum_{j=\frac{n-1-m}{2}+3}^{\frac{n-1}{2}} x^{m-3+j} + 2\sum_{i=2}^{\frac{m}{2}} x^{n-3+i} + 2\sum_{i=3}^{\frac{m}{2}} x^{\frac{n-1}{2}+1+m-i}. \blacksquare$$

The following results are direct consequences of the Theorems 2.1 and 2.2.

Corollary 2.5 For even $m \geq 6$

$$D^*(\theta(m,m);x) = 2D^*(C_m;x) + 2\sum_{i=2}^{\frac{m}{2}}\sum_{j=2}^{\frac{m}{2}} x^{2m-(i+j)} + 2\sum_{i=3}^{\frac{m}{2}}\sum_{j=3}^{\frac{m}{2}} x^{2m+1-(i+j)} + 4\sum_{j=3}^{\frac{m}{2}} x^{m-3+j} - x + 2x^{m-1} - 2. \blacksquare$$

Corollary 2.6 For odd $m \geq 7$

$$D^*(\theta(m,m);x) = 2D^*(C_m;x) + 2\sum_{i=2}^{\frac{m-1}{2}}\sum_{j=2}^{\frac{m-1}{2}} x^{2m-(i+j)} + 2\sum_{i=3}^{\frac{m-1}{2}}\sum_{j=3}^{\frac{m-1}{2}} x^{2m+1-(i+j)} + 4\sum_{i=3}^{\frac{m-1}{2}} x^{m-3+i} + 4\sum_{j=3}^{\frac{m-1}{2}} x^{\frac{3m+1}{2}-j} + 4x^{\frac{3m-5}{2}} + 2x^{m-1} + x^m - x - 2\,. \blacksquare$$

3 The Restricted Detour Index of the Theta Graph

The **detour index** dd^*(G) of a connected graph G is the Wiener index with respect to the restricted detour distance, that is

$$dd^*(G) = \sum_{u,v} D^*(u,v),$$

where the summation is taken over all unordered pairs u, v of vertices of the graph G[2].

It is clear that dd^*(G)= $\frac{d}{dx} D^*(G;x)|_{x=1}$.

Taking the derivatives of $D^*(\theta(m,n);x)$ given in the results in Section 2 at $x = 1$, we get the restricted detour index of the theta graph $\theta(m,n)$ as is given in the next corollary.

Corollary 3.1

(1) For even $m,n \geq 6$, we have

$$dd^*\big(\theta(m,n)\big) = \frac{3}{8}(m+n)\left(m^2+n^2+mn+\frac{80}{3}\right) - 3(m^2+n^2) - \frac{11}{2}mn - 11 .$$

(2) For odd $m,n \geq 7$, we have

$$dd^*\big(\theta(m,n)\big) = \frac{3}{8}(m+n)\left(m^2+n^2+mn+\frac{83}{3}\right) - 3(m^2+n^2) - \frac{11}{2}mn - \frac{27}{2}$$

(3) For odd $m \geq 7$ and even $n \geq 8$, we have

$$dd^*\big(\theta(m,n)\big) = \frac{3}{8}(m^3+n^3) - 3(m^2+n^2) + \frac{3}{4}(m^2n+n^2m) + \frac{81}{8}m + \frac{41}{4}n - \frac{11}{2}mn - \frac{23}{2}$$

(4) For even $m \geq 6$ and odd $n \geq 7$, we have

$$dd^*\big(\theta(m,n)\big) = \frac{3}{8}(m^3+n^3) - 3(m^2+n^2) + \frac{3}{4}(m^2n+n^2m) + \frac{81}{8}n + \frac{41}{4}m - \frac{11}{2}mn - \frac{23}{2} .$$

(5) For even $m \geq 6$ we have

$$dd^*\big(\theta(m,m)\big) = \frac{9}{4}m^3 - \frac{23}{2}m^2 + 20m - 11 .$$

(6) For odd $m \geq 7$, we have

$$dd^*\big(\theta(m,m)\big) = \frac{9}{4}m^3 - \frac{23}{2}m^2 + \frac{83}{4}m - \frac{27}{2} .$$

Proof. Obvious.■

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