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Hosoya Polynomial, Wiener Index, Coloring and Planar of Annihilator Graph of Z_n

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ABSTRACT

Let R be a commutative ring with identity. We consider $\Gamma_{\mathrm{B}}(R)$ an annihilator graph of the commutative ring R. In this paper, we find Hosoya polynomial, Wiener index, Coloring, and Planar annihilator graph of Z_n denote $\Gamma_{\mathrm{B}}(Z_n)$, with $n = p^m$ or $n = p^m q$, where p,q are distinct prime numbers and m is an integer with $m \geq 1$.

Keywords: Annihilator graph of ring, Zero-divisor graph, Hosoya polynomial, Wiener index, coloring graph and planar graph.

متعددة حدود هوسويا، دليل وينر، التلوين والاستواء لبيان تالف الحلقة الإبدالية Z_n

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الملخص

لتكن R حلقة إبدالية بعنصر محايد. نعتبر $\Gamma_{\mathrm{B}}(R)$ بيان لتالف R. في هذا البحث، وجدنا متعددة حدود هوسويا، دليل وينر، التلوين والاستواء لبيان تالف Z_n نرمز له $\Gamma_{\mathrm{B}}(Z_n)$، حيث $n = p^m$ أو $n = p^m q$ بحيث أن: p و q عددان أوليان مختلفان و m عدد صحيح أكبر أو يساوي واحد.

الكلمات المفتاحية: بيان تالف الحلقة، بيان القواسم الصفرية، متعددة حدود هوسويا، دليل وينر، تلوين البيان واستواء البيان.

1. Introduction

Let R be a commutative ring with identity the annihilator of R is the set of all element $x \in R$ satisfy $ann(R) = \{x \in R: x.y = 0 , \forall\, y \in R\}$ [6], and let $ann(R)$ be the set of all annihilator in R. We consider a simple graph $\Gamma_{\mathrm{B}}(R)$ to R with vertices $ann(R)$, for every two distinct vertices x,y are adjacent if and only if $\{x.y = 0: x, y \in ann(R)\}$, and let $Z(R)$ be the set of all zero-divisors in R, and $Z(R)^*$ is the set of all non-zero zero-divisors in it. A simple graph $\Gamma(R)$ with vertices $Z(R)^*$, for every two distinct vertices x,y are adjacent if and only if $\{x.y = 0: x, y \in Z(R)^*\}$.

The notion of an annihilator graph of a commutative ring was first introduced in 1988 by Beck [5], where he was interested in colorings, this investigation was then continued by Anderson and Nasser [3] zero-divisor graph of a commutative ring, further that Anderson and Livingston [2]. They denoted that by $\Gamma(R)$. It is clear that from Beck's definition of annihilator graph of a commutative ring and Anderson's definition of a zero-divisor graph of a commutative ring can be defined Annihilator graph of a commutative ring can be defined $\Gamma_{\mathrm{B}}(R) = \big((\Gamma(R) \cup \{ann(R^*) - Z(R)^*\}) + k_1\big)$. Such that: $\Gamma(R)$ zero-divisor graph of the ring, $ann(R^*)$ set of all vertices in R non-zero, $Z(R)^*$ set of all non-zero zero-divisors in R and $k_1 = 0$.

A graph G is called a connected graph if there is at least one path between any pair of vertices in G, otherwise it is called disconnected [7]. For vertices x,y of G, let $d(x, y)$ be the length of the shortest path from x to y (and it is called distance between two vertices x, y in G). The maximum distance between any two vertices x, y in G is called the diameter graph G [7], that is $diam(G) = \max_{x,y\in V(G)} \{d(x, y)\}$, where $V(G)$ is the set of all vertices of G. A graph G is complete if every two of its vertices are adjacent, so the complete graph of order n is denoted by k_n. If the vertex set of a graph G can be split into two disjoint sets A and B (such that the induced subgraph that generated by either A or B is null graph), then we said G is a bipartite graph. This graph is also said to be a complete bipartite graph is a bipartite graph in graph if each vertex in the set A has joined to every vertex in the set B with just one edge.

Hosoya polynomial of the graph G is defined by $H(G; x) = \sum_{k=0}^{diam(G)} d(G, k)x^k$, where $d(G, k)$ is the number of pairs of vertices of a graph G, that are at distance k apart, for $k = 0,1,2, \dots, diam(G)$. The Wiener index of G is defined as the sum of all distances between vertices of the graph G, and denoted by $W(G)$, we can also find this index by differentiating Hosoya polynomial with respect to x at x =1, by symbols we can write: $W(G) = \frac{d}{dx} H(G; x)\Big|_{x=1}$, See [8,12].

Let $\mathcal{X}(G)$ denote the chromatic number of vertices, i.e., the minimal number of colors, which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors [7]. We let $\breve{\mathcal{X}}(G)$ denote the chromatic number of edges, i.e., the minimal number of colors, which can be assigned to the edges of G in such a way that every two adjacent edges have different colors [7]. And last we assumed $f(G)$ denote the chromatic number of faces, i.e., the minimal number of colors, that can be assigned to the faces of planar graph G in such a way that every two adjacent faces have different colors [7]. A planar graph G is a graph that can be drawn in the plane without crossings for any two edges in G [7]. There are many studies in the graph properties and commutative ring. See [1],[4],[10]&[11].

2. Some Properties of Graph $\Gamma_{\mathrm{B}}(Z_{p^m})$

We will start this item by a lemma.

Lemma 2.1: The vertex (0) connect with every vertex of the graph $\Gamma_{\mathrm{B}}(Z_n)$.
Proof: Since $0.a = 0$, $\forall\, a \in Z_n$, so it is the vertex (0) connect with every vertex of the graph $\Gamma_{\mathrm{B}}(Z_n)$.

Lemma 2.2 [7]: Let G be a connected graph of order p, then:

$$\sum_{k=0}^{diam(G)} d(G, k) = \binom{p+1}{2} = \frac{1}{2}p(p + 1).$$

Theorem 2.3: The Hosoya polynomial of graph $\Gamma_B(Z_{p^m})$ where p is a prime number and m is an integer with $m \geq 1$.

$$H(\Gamma_B(Z_{p^m});x) = p^m + \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]x + \frac{1}{2}\left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]x^2.$$

Proof: From the definition of the graph $\Gamma_B(R)$**,** since the vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$, so the order of the graph $\Gamma_B(Z_n)$ which represents absolute term of Hosoya polynomial of the graph $\Gamma_B(Z_{p^m})$.

Now, we find the coefficient of x that represent size of the graph $\Gamma_B(Z_{p^m})$ using the definition of the graph $\Gamma_B(R)$ is the sum of $(p^m - 1)$ of the edges (since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m})$ from the Lemma (2.1), with a_1 of the graph $\Gamma(Z_{p^m})$ [9] where as $\left(a_1 = \frac{1}{2}\left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2\right]\right)$ so we get.

$$a_1 + (p^m - 1) = \frac{1}{2}\left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2\right] + (p^m - 1)$$
$$= \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right].$$

Now, we find the coefficient of x^2 as the diameter of the graph $\Gamma_B(Z_{p^m})$ is two from the Lemma (2.1) and using Lemma (2.2) so we get:

$$\sum_{k=0}^{\mathrm{diam}(\Gamma_B(Z_{p^m}))} d(\Gamma_B(Z_{p^m}),k) = \binom{p^m + 1}{2}$$
$$\Longrightarrow \frac{p^m(p^m+1)}{2} = d(\Gamma_B(Z_{p^m}),0) + d(\Gamma_B(Z_{p^m}),1) + d(\Gamma_B(Z_{p^m}),2)$$
$$d(\Gamma_B(Z_{p^m}),2) = \frac{p^m(p^m+1)}{2} - d(\Gamma_B(Z_{p^m}),0) - d(\Gamma_B(Z_{p^m}),1)$$
$$= \frac{p^m(p^m+1)}{2} - p^m - \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]$$
$$= \frac{1}{2}\left[p^{2m} + p^m - 2p^m - mp^m - p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]$$
$$= \frac{1}{2}\left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]. \blacksquare$$
$$\therefore H(\Gamma_B(Z_{p^m});x) = p^m + \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]x + \frac{1}{2}\left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]x^2.$$

Corollary 2.4: The Wiener index of graph $\Gamma_B(Z_{p^m})$ where p is prime number and m is an integer with $m \geq 1$.

$$W(\Gamma_B(Z_{p^m})) = \frac{1}{2}\left[2p^{2m} - (m+3)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right].$$

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\because W(\Gamma_B(Z_{p^m})) = \frac{d}{dx}H(\Gamma_B(Z_{p^m});x)\Big|_{x=1}$$
$$\therefore W(\Gamma_B(Z_{p^m})) = \frac{d}{dx}\left(p^m + \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]x + \frac{1}{2}\left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]x^2\right)\Big|_{x=1}$$
$$= 0 + \frac{1}{2}\left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]$$

$$+\left[p^{2m}-(m+2)p^{m}+mp^{m-1}+p^{\left\lfloor\frac{m}{2}\right\rfloor}\right]x\Big|_{x=1}$$

$$=\frac{1}{2}\Big[mp^{m}+p^{m}-mp^{m-1}-p^{\left\lfloor\frac{m}{2}\right\rfloor}+2p^{2m}-2mp^{m}-4p^{m}$$

$$+2mp^{m-1}+2p^{\left\lfloor\frac{m}{2}\right\rfloor}\Big]$$

$$=\frac{1}{2}\left[2p^{2m}-(m+3)p^{m}+mp^{m-1}+p^{\left\lfloor\frac{m}{2}\right\rfloor}\right].\ \blacksquare$$

Example 1: The Hosoya polynomial and wiener index of graph $\Gamma_{\mathrm{B}}(Z_{16})$.
The graph is clear $\Gamma_{\mathrm{B}}(Z_{16})$ of formula $\Gamma_{\mathrm{B}}(Z_{p^m})$, where $p=2$ and $m=4$.

$$\because H(\Gamma_{\mathrm{B}}(Z_{p^m});x)=p^{m}+\frac{1}{2}\left[(m+1)p^{m}-mp^{m-1}-p^{\left\lfloor\frac{m}{2}\right\rfloor}\right]x$$

$$+\frac{1}{2}\left[p^{2m}-(m+2)p^{m}+mp^{m-1}+p^{\left\lfloor\frac{m}{2}\right\rfloor}\right]x^{2}.$$

$$\therefore H(\Gamma_{\mathrm{B}}(Z_{16});x)=16+22x+98x^{2}.$$

$$\because W(\Gamma_{\mathrm{B}}(Z_{p^m}))=\frac{1}{2}\left[2p^{2m}-(m+3)p^{m}+mp^{m-1}+p^{\left\lfloor\frac{m}{2}\right\rfloor}\right].$$

$$\therefore\ \mathrm{W}(\Gamma_{\mathrm{B}}(\mathrm{Z}_{16}))=218.$$

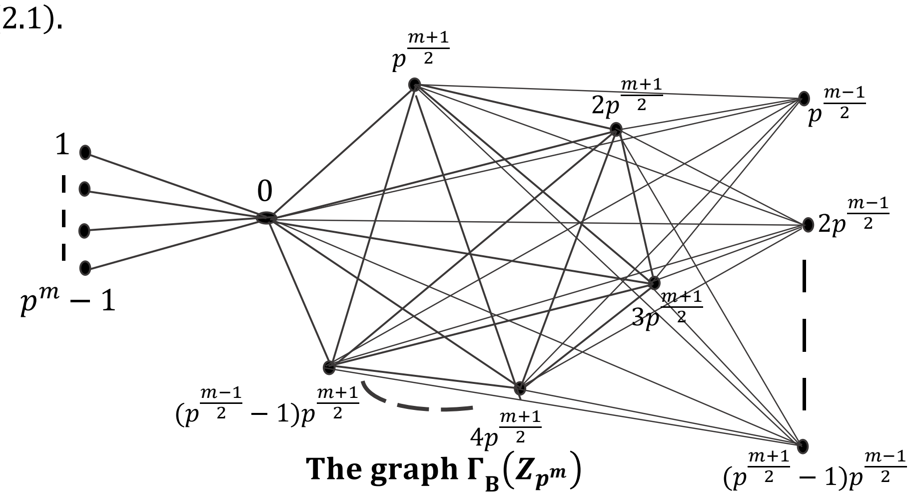
Theorem 2.5: $\left(\textbf{Coloring of graph } \boldsymbol{\Gamma_{\mathrm{B}}(Z_{p^m})}\right)$.

A- Chromatic number of vertices of the graph $\Gamma_{\mathrm{B}}(Z_{p^m})=\begin{cases}p^{\frac{m-1}{2}}+1 & ,m \text{ is an odd.}\\ p^{\frac{m}{2}} & ,m \text{ is an even.}\end{cases}$

B- Chromatic number of edges of the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ is $p^{m}-1$.

Proof: A- Case 1: if m is an odd.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ by one of its complication $\left(2p^{\frac{m+1}{2}},3p^{\frac{m+1}{2}},\dots,p^{\frac{m-1}{2}}.p^{\frac{m+1}{2}}=0\right)$, that the product is one of its complications of the number p^{m} which is equal to (0) in the ring $\boldsymbol{Z_{p^m}}$**.** Or multiplication one complication of the number $p^{\frac{m+1}{2}}$ in another complication of the number $p^{\frac{m+1}{2}}$ that the product is one of the complications of the number p^{m} which is equal to (0) in the ring $Z_{\boldsymbol{p^m}}$ as in the Figure (2.1).

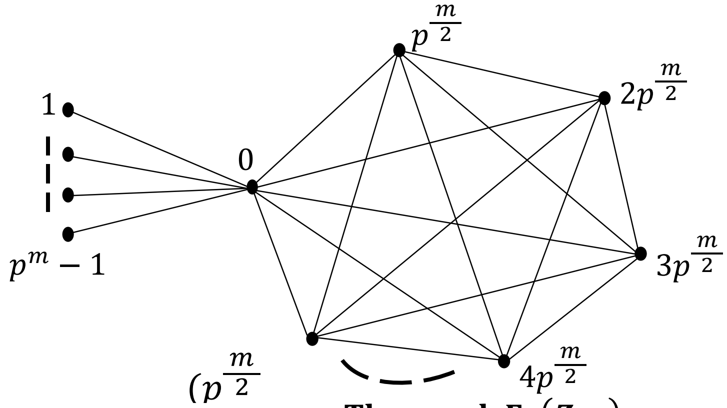


The graph $\boldsymbol{\Gamma_{\mathrm{B}}(Z_{p^m})}$
Fig (2. 1)

Clearly the complete graph $k_{p^{\frac{m-1}{2}}}$ be subgraph from the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ (when m is an odd), and also the multiplication of the number $p^{\frac{m-1}{2}}$ or one of its complications $\left(2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \dots, p^{\frac{m+1}{2}}.p^{\frac{m-1}{2}} = 0\right)$ by the number $p^{\frac{m+1}{2}}$ or one of its complications is the product p^m or one of its complications which is equal to (0), in the ring $Z_{\boldsymbol{p^m}}$. And thus the complete graph $k_{p^{\frac{m-1}{2}}+1}$ is the largest complete subgraph that exist in the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$. And since the chromatic number of vertices of a complete graph $k_{p^{\frac{m-1}{2}}+1}$ is $(p^{\frac{m-1}{2}} + 1)$ [7], so it is the chromatic number of the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ is $(p^{\frac{m-1}{2}} + 1)$ and also of vertices (when m is an odd).

A- Case 2: if m is an even:

Since the multiplication of the number $p^{\frac{m}{2}}$ by one of its complications $\left(2p^{\frac{m}{2}}, 3p^{\frac{m}{2}}, \dots, p^{\frac{m}{2}}.p^{\frac{m}{2}} = 0\right)$, that the product is one of its complications of the number p^m which is equal to (0) in the ring $Z_{\boldsymbol{p^m}}$**.** Or multiplying one complication of the number $p^{\frac{m}{2}}$ in another complication of the number $p^{\frac{m}{2}}$, that the product is one of its complications of the number p^m, which is equal to (0) in the ring $\mathrm{Z}_{\boldsymbol{p^m}}$ as in the Figure (2.2).



The graph $\boldsymbol{\Gamma_{\mathrm{B}}(Z_{p^m})}$
Fig (2.2)

Clearly the complete graph $k_{p^{\frac{m}{2}}}$ is the largest complete subgraph that exist in the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ (when m is an even). And since the chromatic number of vertices of a complete graph $\mathrm{k}_{p^{\frac{m}{2}}}$ is $p^{\frac{m}{2}}$ [7], so it is the chromatic number of the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ is $p^{\frac{m}{2}}$ and also of vertices (when m is an even).

B- From the Lemma (2.1) the vertex (0) connect with every vertex in the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ then the degree of the vertex (0) is $(p^m - 1)$ so the chromatic number of the edges is $(p^m - 1)$.

Theorem 2.6 [7]: (kuratowski's Theorem), The graph G is planar if and only if it does not contain G on subgraph that is homeomorphic to k_5 or $k_{3,3}$.

Theorem 2.7:

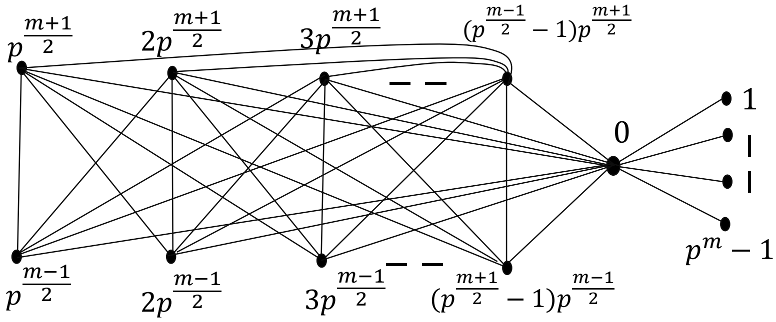
a- The graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ contains a subgraph that is homeomorphic to $k_{p^{\frac{m-1}{2}}+1}$ and $k_{\left(p^{\frac{m+1}{2}}-p^{\frac{m-1}{2}}\right),p^{\frac{m-1}{2}}}$ (when m is an odd).

b- The graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ contains a subgraph that is homeomorphic to $k_{p^{\frac{m}{2}}}$ and $k_{\left(p^{\frac{m+2}{2}}-p^{\frac{m-2}{2}}\right),p^{\frac{m-2}{2}}}$ (when m is an even).

Proof:

a- From the Theorem (2.5-A-1) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ or one of its complications $\left(2p^{\frac{m+1}{2}}, 3p^{\frac{m+1}{2}}, \dots, p^{\frac{m-1}{2}}.p^{\frac{m+1}{2}} = 0\right)$ by number $p^{\frac{m-1}{2}}$ or one of its complications $\left(2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \dots, p^{\frac{m+1}{2}}.p^{\frac{m-1}{2}} = 0\right)$ be the product p^m or one of its complications of the number p^m which is equal to (0) in the ring $Z_{\boldsymbol{p^m}}$. Thus, the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ contains a subgraph homeomorphic with complete bipartite graph $k_{\left(p^{\frac{m+1}{2}}-p^{\frac{m-1}{2}}\right),p^{\frac{m-1}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ as in the Figure (2.3).

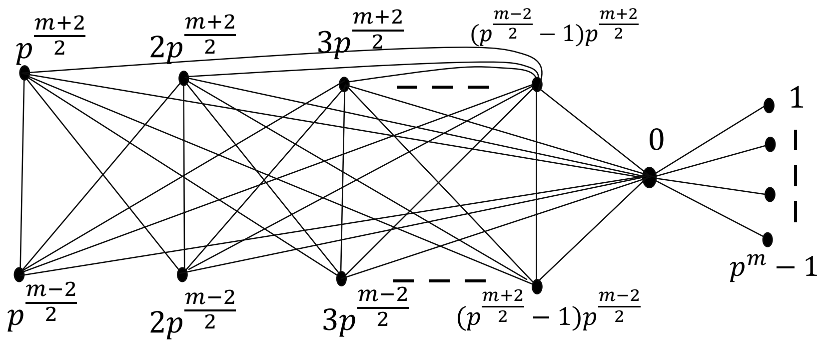


The graph $\boldsymbol{\Gamma_{\mathrm{B}}(Z_{p^m})}$

Fig (2.3)

b- From the Theorem (2.5-A-2) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+2}{2}}$ or one of its complications $\left(2p^{\frac{m+2}{2}}, 3p^{\frac{m+2}{2}}, \dots, p^{\frac{m-2}{2}}.p^{\frac{m+2}{2}} = 0\right)$ by number $p^{\frac{m-2}{2}}$ or one of its complications $\left(2p^{\frac{m-2}{2}}, 3p^{\frac{m-2}{2}}, \dots, p^{\frac{m+2}{2}}.p^{\frac{m-2}{2}} = 0\right)$ be the product p^m or one of its complications of the number p^m which is equal to (0) in the ring $Z_{\boldsymbol{p^m}}$. Thus, the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ contains a subgraph homeomorphic complete bipartite graph $k_{\left(p^{\frac{m+2}{2}}-p^{\frac{m-2}{2}}\right),p^{\frac{m-2}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_{\mathrm{B}}(Z_{p^m})$ as in the Figure (2.4).



The graph $\Gamma_B(Z_{p^m})$
Fig (2.4)

Remarks:

1. From the Theorem (2.7-a), the only graphs $\Gamma_B(Z_p)$ and $\Gamma_B(Z_8)$ from the formula $\Gamma_B(Z_{p^m})$ when m is an odd it does not contain subgraph homeomorphic k_5 or $k_{3,3}$ therefore it is planar graphs by kuratowski's Theorem.
2. From the Theorem (2.7-b), the only graphs $\Gamma_B(Z_4)$, $\Gamma_B(Z_9)$ and $\Gamma_B(Z_{16})$ from the formula $\Gamma_B(Z_{p^m})$ when m is an even it does not contain subgraph homeomorphic k_5 or $k_{3,3}$ therefore it is planar graphs by kuratowski's Theorem.
3. The only graphs $\Gamma_B(Z_4)$, $\Gamma_B(Z_8)$, $\Gamma_B(Z_9)$, $\Gamma_B(Z_{16})$ and $\Gamma_B(Z_p)$ they are colorable for faces.

Example 2: The chromatic number of the graphs $\Gamma_B(Z_{16})$ and $\Gamma_B(Z_{27})$.

The graph is clear $\Gamma_B(Z_{16})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 2$ and $m = 4$ and the graph is clear $\Gamma_B(Z_{27})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 3$ and $m = 3$.

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m}{2}}$ (when m is an even).

$\therefore \mathcal{X}(\Gamma_B(Z_{16})) = 4.$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

$\therefore \breve{\mathcal{X}}(\Gamma_B(Z_{16})) = 15.$

From Theorem (2.7-b) we get the graph $\Gamma_B(Z_{16})$ contains a subgraph that is homeomorphic to k_4 and $k_{6,2}$ then the graph $\Gamma_B(Z_{16})$ it is planar by kuratowski's Theorem.

$\therefore f(\Gamma_B(Z_{16})) = 3.$

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m-1}{2}} + 1$ (when m is an odd).

$\therefore \mathcal{X}(\Gamma_B(Z_{27})) = 4.$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

$\therefore \breve{\mathcal{X}}(\Gamma_B(Z_{27})) = 26.$

From Theorem (2.7-a) we get the graph $\Gamma_B(Z_{27})$ contains a subgraph that is homeomorphic to $k_{6,3}$ then the graph $\Gamma_B(Z_{27})$ it is not planar by kuratowski's Theorem.

3. Some Properties of graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$.

Theorem 3.1: The Hosoya polynomial of graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ where p, q are distinct prime numbers and m is an integer with $m \geq 1$.

$$H(\Gamma_{\mathrm{B}}(Z_{p^m q}); x) = p^m q + \left(\frac{1}{2}[2q(mp - m + p) - (m+1)p + m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x + \left(\frac{1}{2}[q(p^{m+1}q - 3p - 2mp + 2m) + (m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x^2.$$

Proof: From the definition of the graph $\Gamma_{\mathrm{B}}(R)$ since the vertex (0) connect with every vertex the graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ so the order of the graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ which represents absolute term Hosoya polynomial of graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$.

Now, we find the coefficient of x that represent size of the graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ using the definition of the graph $\Gamma_{\mathrm{B}}(R)$ is the sum of $(Z_{p^m q} - 1)$ of the edges (since the vertex (0) connect with every vertex the graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ from the Lemma (2.1), with a_1 of the graph $\Gamma(Z_{p^m q})$ [9] where as $\left(a_1 = \frac{1}{2}[2mq(p-1) - (m+1)p + m]p^{m-1} - p^{\left\lfloor\frac{m}{2}\right\rfloor} + 1\right)$ so we get.

$$a_1 + (p^m q - 1) = \frac{1}{2}[2mq(p-1) - (m+1)p + m]p^{m-1} - p^{\left\lfloor\frac{m}{2}\right\rfloor} + 1 + p^m q - 1$$
$$= \frac{1}{2}[2q(mp - m + p) - (m+1)p + m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}.$$

Now, we find the coefficient of x^2 as the diameter of the graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ is two from the Lemma (2.1) and using Lemma (2.2) so we get.

$$\sum_{k=0}^{\operatorname{diam}(\Gamma_B(Z_{p^m q}))} d(\Gamma_{\mathrm{B}}(Z_{p^m q}), k) = \binom{p^m q + 1}{2}$$
$$\Longrightarrow \frac{p^m q(p^m q + 1)}{2} = d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 0) + d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 1) + d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 2)$$
$$d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 2) = \frac{p^m q(p^m q + 1)}{2} - d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 0) - d(\Gamma_{\mathrm{B}}(Z_{p^m q}), 1)$$
$$= \frac{p^m q(p^m q + 1)}{2} - p^m q - \left(\frac{1}{2}[2q(mp - m + p) + m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right).$$
$$= \frac{1}{2}[q(p^{m+1}q - 3p - 2mp + 2m) + (m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}. \blacksquare$$
$$\therefore H(\Gamma_{\mathrm{B}}(Z_{p^m q}); x) = p^m q + \left(\frac{1}{2}[2q(mp - m + p) - (m+1)p + m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x + \left(\frac{1}{2}[q(p^{m+1}q - 3p - 2mp + 2m) + (m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x^2.$$

Corollary 3.2: The Wiener index of $\Gamma_{\mathrm{B}}(Z_{p^m q})$ where p, q are distinct prime numbers and m is an integer with $m \geq 1$.

$$W\left(\Gamma_{\mathrm{B}}(Z_{p^m q})\right) = \frac{1}{2}[2q(p^{m+1}q - mp - 2p + m) + (m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}$$

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\because W(\Gamma_{\mathrm{B}}(Z_{p^m q})) = \frac{d}{dx}H(\Gamma_B(Z_{p^m q}); x)\Big|_{x=1}$$
$$\therefore W(\Gamma_{\mathrm{B}}(Z_{p^m q})) = \frac{d}{dx}\left(p^m q + \left(\frac{1}{2}[2q(mp - m + p) - (m+1)p + m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x + \left(\frac{1}{2}[q(p^{m+1}q - 3p - 2mp + 2m) + (m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x^2\right)\Big|_{x=1}$$

$$= \left(0 + \left(\frac{1}{2}[2q(mp-m+p)-(m+1)p+m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right) + \left([q(p^{m+1}q-3p-2mp+2m)+(m+1)p-m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x\right|_{x=1}$$

$$= \frac{1}{2}[2q(p^{m+1}q-mp-2p+m)+(m+1)p-m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}. \blacksquare$$

Example 3: The Hosoya polynomial and wiener index of graph $\Gamma_B(Z_{18})$.

The graph is clear $\Gamma_B(Z_{18})$ of formula $\Gamma_B(Z_{p^m q})$, where $p = 3$, $q = 2$ and $m = 2$.

$\because H(\Gamma_B(Z_{p^m q}); x) = p^m q + \left(\frac{1}{2}[2q(mp-m+p)-(m+1)p+m]p^{m-1} - \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x + \left(\frac{1}{2}[2q(p^{m+1}q-mp-2p+m)+(m+1)p - m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}\right)x^2$

$\therefore H(\Gamma_B(Z_{18}); x) = 18 + 30x + 123x^2$.

$\because W\left(\Gamma_B(Z_{p^m q})\right) = \frac{1}{2}[2q(p^{m+1}q-mp-2p+m)+(m+1)p-m]p^{m-1} + \frac{1}{2}p^{\left\lfloor\frac{m}{2}\right\rfloor}$.

$\therefore$ $W(\Gamma_B(Z_{18})) = 276$.

Theorem 3.3: $\left(\textbf{Coloring of graph } \boldsymbol{\Gamma_B(Z_{p^m q})}\right)$.

A- Chromatic number of vertices of the graph

$$\Gamma_B(Z_{p^m q}) = \begin{cases} p^{\frac{m-1}{2}} + 2 & , m \text{ is an odd.} \\ p^{\frac{m}{2}} + 1 & , m \text{ is an even.} \end{cases}$$

B- Chromatic number of edges of the graph $\Gamma_B(Z_{p^m q})$ is $p^m q - 1$.

Proof: A- Case 1: if m is an even:

From the Theorem (2.5-A-1). Since the subgraph $k_{p^{\frac{m-1}{2}}+1}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m})$ (when m is an odd). It is also clear that the number $p^m q$ product of multiplication the number p^m or one of its complications in the number q or one of its complications thus a new vertex will be added to the complete graph $k_{p^{\frac{m-1}{2}}+1}$ so we have the complete graph $k_{p^{\frac{m-1}{2}}+2}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m q})$ hence the chromatic number of the graph $\Gamma_B(Z_{p^m q})$ is $\left(p^{\frac{m-1}{2}} + 2\right)$ [7].

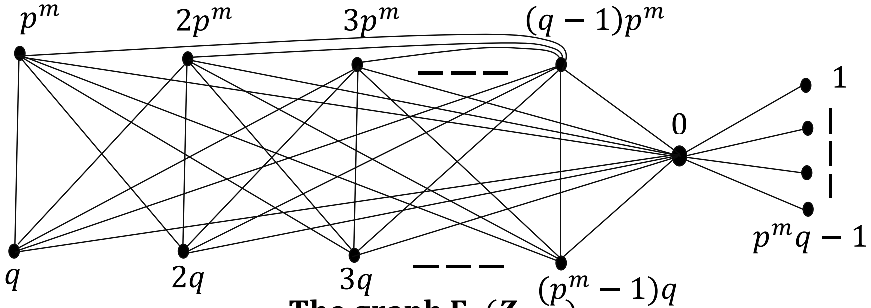
A- Case 2: if m is an even:

From the Theorem (2.5-A-2). Since the subgraph $k_{p^{\frac{m}{2}}}$ is the largest complete subgraph, exist in the graph $\Gamma_B(Z_{p^m})$ (when m is an even). It is also clear that the number $p^m q$ product of multiplication the number p^m or one of its complications in the number q or one of its complications thus a new vertex will be added to the complete graph $k_{p^{\frac{m}{2}}}$ so we have the complete graph $k_{p^{\frac{m}{2}}+1}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m q})$ hence the chromatic number of the graph $\Gamma_B(Z_{p^m q})$ is $\left(p^{\frac{m}{2}} + 1\right)$ [7].

B- From the Lemma (2.1) so it is the vertex (0) connect with every vertex the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ then the degree of the vertex (0) is $(p^m q-1)$ so it is the chromatic number of the edges is $(p^m q-1)$.

Theorem 3.4: The graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$ contains a subgraph that is homeomorphic to $k_{(p^m-1),q}$ and $k_{(p.q-p^{m-1}),p^{m-1}}$.

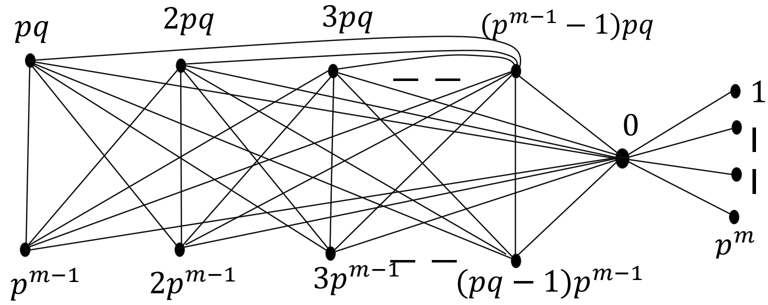
Proof: The first part, since the multiplying the number p^m or one of its complications $(2p^m, 3p^m, \dots, (q-1).p^m, p^m q = 0)$ by number q or one of its complications $(2q, 3q, \dots, (p^m-1).q, p^m q = 0)$ be the product $p^m q$ or one a complications of the number $p^m q$ which is equal to (0) in the ring $Z_{p^m q}$. Thus the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ contains a subgraph homeomorphic complete bipartite graph $k_{(p^m-1),q}$, as in the Figure (3.1).



The graph $\Gamma_{\mathrm{B}}(Z_{p^m q})$

Fig (3.1)

The second part, since the multiplying the number pq or one of its complications $(2pq, 3pq, \dots, (p^{m-1}-1)pq, p^m q = 0)$ by number p^{m-1} or one of its complications $(2p^{m-1}, 3p^{m-1}, \dots, (pq-1)p^{m-1}, p^m q = 0)$ be the product $p^m q$ or one a complications of the number $p^m q$ which is equal to (0) in the ring $Z_{p^m q}$. Thus the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ contains a subgraph homeomorphic complete bipartite graph $k_{(pq-p^{m-1}),p^{m-1}}$, as in the Figure (3.2).



The graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$

Fig (3.2)

Remark:

From the Theorem (3.4), the only graphs of the formula $\Gamma_{\mathrm{B}}(Z_{p^m q})$ when $q=2$ and $m=1$ does not contain a subgraph homeomorphic $k_{3,3}$ or k_5 therefore it is planar and colorable for faces. Otherwise, the graphs of the formula $\Gamma_{\mathrm{B}}(Z_{p^m q})$ contain a subgraph homeomorphic $k_{3,3}$ or k_5 therefore it is not planar graphs by kuratowski's Theorem.

Example 4: The chromatic number of the graphs $\Gamma_{\mathrm{B}}(Z_{18})$ and $\Gamma_{\mathrm{B}}(Z_{22})$

The graph is clear $\Gamma_{\mathrm{B}}(Z_{18})$ of formula $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$, where $p = 3\,, q = 2$ and $m = 2$ and the graph is clear $\Gamma_{\mathrm{B}}(Z_{22})$ of formula $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$, where $p = 11\,, q = 2$ and $m = 1$.

The chromatic number of vertices of the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ is $p^{\frac{m}{2}} + 1$ (when m is an even).

$\therefore \mathcal{X}(\Gamma_{\mathrm{B}}(Z_{18})) = 4$.

The chromatic number of edges the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ is $p^m q - 1$.

$\therefore \breve{\mathcal{X}}(\Gamma_{\mathrm{B}}(Z_{18})) = 17$.

From Theorem (3.4) we get the graph $\Gamma_{\mathrm{B}}(Z_{18})$ contains a subgraph that is homeomorphic to $k_{3,3}$ then the graph $\Gamma_{\mathrm{B}}(Z_{18})$ it is not planar by kuratowski's Theorem.

The chromatic number of vertices the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ is $p^{\frac{m-1}{2}} + 2$ (when m is an odd).

$\therefore \mathcal{X}(\Gamma_{\mathrm{B}}(Z_{22})) = 3$.

The chromatic number of edges the graph $\Gamma_{\mathrm{B}}\left(Z_{p^m q}\right)$ is $p^m q - 1$.

$\therefore \breve{\mathcal{X}}(\Gamma_{\mathrm{B}}(Z_{22})) = 21$.

From Theorem (3.4) we get the graph $\Gamma_{\mathrm{B}}(Z_{22})$ contains a subgraph that is homeomorphic to $k_{10,2}$ it is the largest complete bipartite graph there is in the graph $\Gamma_{\mathrm{B}}(Z_{22})$ then the graph $\Gamma_{\mathrm{B}}(Z_{22})$ it is planar by kuratowski's Theorem.

$\therefore f(\Gamma_{\mathrm{B}}(Z_{22})) = 3$.

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