Open Access
2020, 17(1):166-171

Baghdad Science Journal

P-ISSN: 2078-8665
E-ISSN: 2411-7986

DOI: http://dx.doi.org/10.21123/bsj.2020.17.1.0166

A New Two Derivative FSAL Runge-Kutta Method of Order Five in Four Stages

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Received 4/10/2018, Accepted 11/6/2019, Published 1/3/2020

Abstract:

A new efficient Two Derivative Runge-Kutta method (TDRK) of order five is developed for the numerical solution of the special first order ordinary differential equations (ODEs). The new method is derived using the property of First Same As Last (FSAL). We analyzed the stability of our method. The numerical results are presented to illustrate the efficiency of the new method in comparison with some well-known RK methods.

Key words: Explicit methods, FSAL property, Initial value problems, Two Derivative Runge-Kutta method.

Introduction:

In the last years, several methods have been proposed to solve numerically first order ODEs of the form

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 = a \le x \le b. \quad (1)$$

where $y \in \mathbb{R}^d$ and $f: \mathbb{R}^d \to \mathbb{R}^d$.

Butcher (1) derived the order conditions of Runge-Kutta method using the theory of trees. Franco (2) constructed the embedded EFRKN4(3) methods relying on FSAL technique. Van de Vyver (3) constructed embedded phase-fitted modified Runge-Kutta method of order five and four based on the FSAL technique for solving the radial Schrödinger equation. Fang et al. (4) developed a new fifth-order Runge-Kutta method and embedded RK5(4) pair based on FSAL technique adapted for solving oscillatory problems. Chan and Tsai (5) constructed an explicit Two Derivative Runge-Kutta (TDRK) methods of order up to seven that include one function evaluation of f and a minimal number of function evaluations of g. They derived the order conditions of TDRK method based on Butcher's theory of trees in (1). Fang et al. (6) proposed fourth-order extended Runge-Kutta Nystrom (EKRN) methods and then they derived embedded EKRN4(3) pairs based on FSAL property to solve second-order ODEs with perturbed oscillators solutions. Very recently, Ahmad and Senu (7) have proposed a new explicit TDRK method of order four based on the FSAL property for solving first order ODEs.

Here in this paper, motivated by Chan and Tsai (5) and Ahmad and Senu (7), we developed a new four-stage fifth-order TDRK method designed utilizing the FSAL technique. The preliminaries of TDRK methods are presented in Section 2. In Section 3, a new TDRK method with FSAL property is derived. In Section 4, we analyzed the stability of the proposed method. In Section 5, numerical tests are given to demonstrate the efficiency of our TDRK method when it is compared with other Runge-Kutta methods in the scientific literature. Finally, in Section 6, we give some conclusion.

Preliminaries:

In this work, we are interested in the efficient numerical method for solving first-order ordinary differential equations (ODEs) (1). We consider the special explicit TDRK methods studied in (5)

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$$\begin{cases} Y_i = y_n + c_i h f(x_n, y_n) + h^2 \sum_{j=1}^{i-1} a_{ij}\, g(x_n + c_j h, Y_j), \quad i = 2, \dots, s \\ y_{n+1} = y_n + h f(x_n, y_n) + h^2 \sum_{i=1}^{s} b_i\, g(x_n + c_i h, Y_i), \end{cases} \tag{2}$$

where
$y''(x) = g(x, y) = f_x(x, y) + f_y(x, y) f(x, y)$. The special explicit TDRK method (2) includes only one function evaluation of f and s function evaluations of g per step. The coefficients of the special explicit TDRK method (2) can be expressed in Butcher tableau as follows:

$$\begin{array}{c|ccccc} 0 & 0 & & & & \\ c_2 & a_{21} & 0 & & & \\ c_3 & a_{31} & a_{32} & 0 & & \\ \vdots & \vdots & \vdots & & 0 & \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss-1} & \\ \hline & b_1 & b_2 & \dots & b_s & \end{array}$$

According to Chan and Tsai (5), the order conditions for new TDRK method up to five are presented as follows
order 2:

$$\sum_{i=1}^{s} b_i = \frac{1}{2}, \tag{3}$$

order 3:

$$\sum_{i=2}^{s} b_i\, c_i = \frac{1}{6}, \tag{4}$$

order 4:

$$\sum_{i=2}^{s} b_i\, c_i^2 = \frac{1}{12}, \tag{5}$$

order 5:

$$\sum_{i=2}^{s} b_i\, c_i^3 = \frac{1}{20}, \qquad \sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i\, a_{ij} c_j = \frac{1}{120}. \tag{6}$$

In practice, the following Nystrom row assumption is helpful in (8)

$$\sum_{j=1}^{s} a_{ij} = \frac{1}{2} c_i^2, \quad i = 2, \dots, s. \tag{7}$$

A Fifth Order TDRK with FSAL Property:

A new TDRK method with “First Same As Last" (FSAL) property will be derived where

$$b_i = a_{si}, \quad i = 1, \dots, s - 1, \quad \text{and} \quad b_s = 0. \tag{8}$$

The feature of “First Same As Last" (FSAL) technique is that the fourth stage can be reused as the first stage of the next step. Therefore, the efficient number of function evaluations is three per step. According to the Nystrom row assumption (7), we have

$$a_{21} = \frac{c_2^2}{2},$$

$$a_{31} = \frac{c_3^2}{2} - a_{32},$$

In order to construct four-stage fifth-order TDRK method by using FSAL technique, solving the order conditions (3)–(6) simultaneously results in a solution with one free parameter c_3 as follows;

$$b_1 = \frac{1}{12}\, \frac{10c_3^2 - 8c_3 + 1}{c_3(-3 + 5c_3)},$$

$$b_2 = -\frac{-200c_3^3 + 300c_3^2 - 150c_3 + 25}{600c_3^3 - 960c_3^2 + 540c_3 - 108},$$

$$b_3 = \frac{1}{120c_3^3 - 120c_3^2 + 36c_3},$$

$$a_{32} = -\frac{-20c_3^4 + 30c_3^3 - 16c_3^2 + 3c_3}{-6 + 10c_3},$$

$$c_2 = -\frac{3 - 5c_3}{10c_3 - 5}.$$

Choosing $c_3 = \frac{4}{5}$, yields a fifth order TDRK method with FSAL property denoted as TDRK5F, which is given in the following Butcher tableau;

Table 1. The TDRK5F method with FSAL property

$$\begin{array}{c|cccc} 0 & 0 & & & \\ \frac{1}{3} & \frac{1}{18} & 0 & & \\ \frac{4}{5} & -\frac{2}{125} & \frac{42}{125} & 0 & \\ 1 & \frac{5}{48} & \frac{9}{28} & \frac{25}{336} & 0 \\ \hline & \frac{5}{48} & \frac{9}{28} & \frac{25}{336} & 0 \end{array}$$

Stability of TDRK5F method:

In this section, we discuss the stability of TDRK5F method. We consider the following test equation:

$$y' = i\lambda y, \quad \lambda > 0. \tag{9}$$

By applying TDRK method (2) to (9) produces the following difference equation

$$y_{n+1} = M(v) y_n, \quad v = \lambda h,$$

where

$$M(v) = (1 + v^2 b^T (I - v^2 A)^{-1} e) + (v + v^2 b^T (I - v^2 A)^{-1} c)$$

where I is the identity matrix and A is the coefficient of the new TDRK5F with

$$c = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{5} \\ 1 \end{bmatrix}, \quad e = [1 \quad 1 \quad 1 \quad 1], \quad b^T = \begin{bmatrix} \frac{5}{48} & \frac{9}{28} & \frac{25}{336} & 0 \end{bmatrix}.$$

The stability function of TDRK5F method is as follows:

$$M(v) = 1 + v + \frac{1}{2}v^2 + \frac{1}{6}v^3 + \frac{1}{24}v^4 + \frac{1}{120}v^5 + \frac{1}{720}v^6.$$

In Figure 1, the stability regions of the TDRK5F method is plotted.

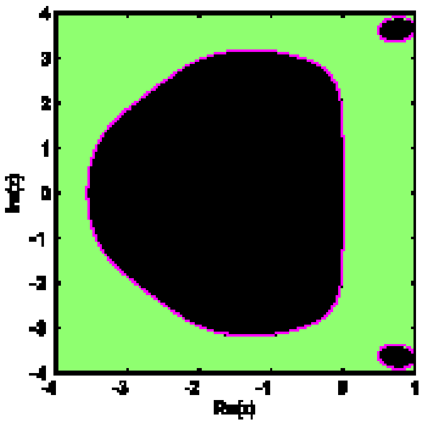

Figure 1. The stability region of TDRK5F method

Numerical Results:

In this section, some test problems is solved using Matlab to show the efficiency of the new TDRK5F method as compared with some efficient RK methods which are selected from the scientific literature. The following methods are used in comparison:

- **TDRK5F:** The new four-stages fifth-order TDRK method with FSAL property constructed in this paper.
- **RK5W:** The five-stages fifth-order RK method given in (9).
- **RK5N:** The six-stages fifth-order RK method given in (10).
- **RK5B:** The six-stages fifth-order RK method given in (8).

The problems are integrated in the interval [0,10]. The accuracy criteria calculated by taking log_{10} of the maximum absolute error as follows:

The accuracy = $\log_{10}(\max \ (|y(x_n) - y_n|))$.

The numerical results and the efficiency curves of the methods are presented in Tables 2, 3,4,5,6 and Figures 2, 3,4,5,6 respectively.

Problem 1: (7)

$$y' = -2xy, \quad y(0) = 1.$$

The analytic solution is: $y(x) = e^{-x^2}$.

Problem 2: (11)

$$y'_1 = y_2,$$
$$y'_2 = -13y_1 + 12y_3 + 9\cos(2x) - 12\sin(2x),$$
$$y'_3 = y_4,$$
$$y'_4 = 12y_1 - 13y_3 - 12\cos(2x) + 9\sin(2x),$$
$$y_1(0) = 1, \quad y_2(0) = -4, \quad y_3(0) = 0, \quad y_4(0) = 8.$$

The analytic solution is:

$$y_1(x) = \sin(x) - \sin(5x) + \cos(2x),$$
$$y_2(x) = \cos(x) - 5\cos(5x) - 2\sin(2x),$$
$$y_3(x) = \sin(x) - \sin(5x) + \sin(2x),$$
$$y_4(x) = \cos(x) - 5\cos(5x) + 2\cos(2x).$$

Problem 3: (Periodic Orbit problem (12))

$$y'_1 = y_2, \quad y'_2 = -y_1 + 0.001\cos(x),$$
$$y'_3 = y_4, \quad y'_4 = y_3 + 0.001\sin(x),$$
$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0.9995.$$

The analytic solution is:

$$y_1(x) = \cos(x) + 0.0005x\sin(x),$$
$$y_2(x) = -0.9995\sin(x) + 0.0005x\cos(x),$$
$$y_3(x) = \sin(x) - 0.0005x\cos(x),$$
$$y_4(x) = 0.9995\cos(x) + 0.0005x\sin(x).$$

Problem 4: (Kepler's problem (13))

$$y'_1 = y_2, \quad y'_2 = -\frac{y_1}{\left(\sqrt{y_1^2 + y_3^2}\right)^3},$$
$$y'_3 = y_4, \quad y'_4 = -\frac{y_3}{\left(\sqrt{y_1^2 + y_3^2}\right)^3},$$
$$y_1(0) = 1 - e, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = \sqrt{\frac{1+e}{1-e}}.$$

The analytic solution is:

$$y_1(x) = \cos(x) - e, \quad y_2(x) = -\sin(x),$$
$$y_3(x) = \sqrt{1 - e^2}\sin(x),$$
$$y_4(x) = \sqrt{1 - e^2}\cos(x).$$

where e $(0 \le e < 1)$ the eccentricity of the orbit, we choose $e = 0$.

Problem 5: (14)

$$y'_1 = y_2,$$
$$y'_2 = -\frac{101}{2}y_1 + \frac{99}{2}y_3 + \frac{93}{2}\cos(2x) - \frac{99}{2}\sin(2x),$$
$$y'_3 = y_4,$$
$$y'_4 = \frac{99}{2}y_1 - \frac{101}{2}y_3 + \frac{93}{2}\sin(2x) - \frac{99}{2}\cos(2x),$$
$$y_1(0) = 0, \quad y_2(0) = -10, \quad y_3(0) = 1, \quad y_4(0) = 12.$$

The analytic solution is:

$$y_1(x) = -\cos(10x) - \sin(10x) + \cos(2x),$$
$$y_2(x) = 10\sin(10x) - 10\cos(10x) - 2\sin(2x),$$
$$y_3(x) = \cos(10x) + \sin(10x) + \sin(2x),$$
$$y_4(x) = -10\sin(10x) + 10\cos(10x) + 2\cos(2x).$$

Table 2. The Numerical Results for Problems 1

h	Method	MAXERR	FC
0.1	TDRK5F	8.260301764817513e-08	401
	RK5W	8.070167282561713e-07	500
	RK5N	4.584971776584734e-07	600
	RK5B	6.939891018763189e-07	600
0.05	TDRK5F	2.426934819776960e-09	801
	RK5W	4.021845012580627e-08	1000
	RK5N	1.258029772369107e-08	1200
	RK5B	1.852752946895908e-08	1200
0.025	TDRK5F	7.354195030728761e-11	1601
	RK5W	2.246330690902632e-09	2000
	RK5N	3.684960774019697e-10	2400
	RK5B	5.354157794207337e-10	2400
0.0125	TDRK5F	2.262079412673757e-12	3201
	RK5W	1.328272214440318e-10	4000
	RK5N	1.114611181129988e-11	4800
	RK5B	1.608168979927438e-11	4800
0.00625	TDRK5F	6.900036098045348e-14	6401
	RK5W	8.076289637060086e-12	8000
	RK5N	3.426946226792182e-13	9600
	RK5B	4.927447339042601e-13	9600

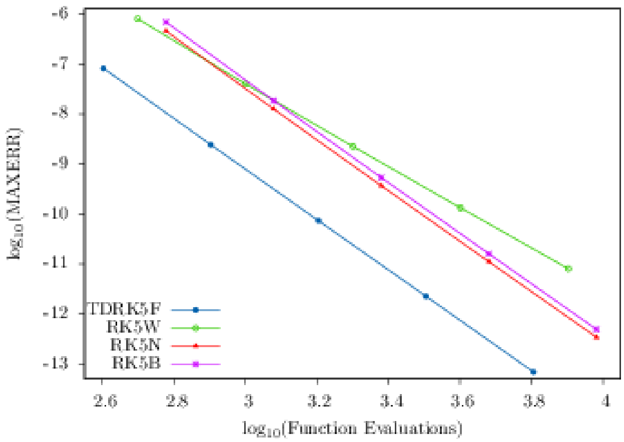

Figure 2. The performance curves with step size $h = 0.\frac{1}{2^i}, i = 0,1,2,3,4.$ for Problem 1.

Table 3. The Numerical Results for Problems 2

h	Method	MAXERR	FC
0.1	TDRK5F	1.179949594860563e-04	401
	RK5W	1.387702855398132e-03	500
	RK5N	1.383163486259886e-03	600
	RK5B	5.005328196120096e-03	600
0.05	TDRK5F	2.141261506577452e-06	801
	RK5W	3.143225541168970e-05	1000
	RK5N	3.118207428803865e-05	1200
	RK5B	1.112072881214732e-04	1200
0.025	TDRK5F	3.519543970154082e-08	1601
	RK5W	7.836249451692590e-07	2000
	RK5N	7.686611112056596e-07	2400
	RK5B	2.699054183652461e-06	2400
0.0125	TDRK5F	5.612864062420897e-10	3201
	RK5W	2.161441428616406e-08	4000
	RK5N	2.069418884864671e-08	4800
	RK5B	7.174273297660960e-08	4800
0.00625	TDRK5F	1.056765785989455e-11	6401
	RK5W	6.514350170405692e-10	8000
	RK5N	5.944259806600627e-10	9600
	RK5B	2.042217006614777e-09	9600

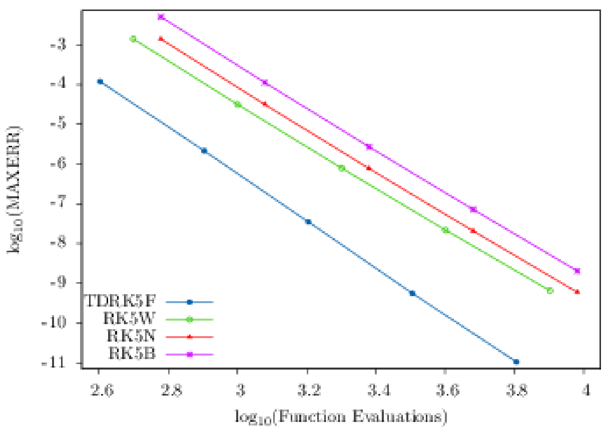

Figure 3. The performance curves with step size $h = 0.1/2^i, i = 0,1,2,3,4.$ for Problem 2.

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

Table 4. The Numerical Results for Problems 3

h	Method	MAXERR	FC
0.125	TDRK5F	6.763564264211652e-09	321
	RK5W	3.775046538700977e-07	400
	RK5N	3.782084007086795e-07	480
	RK5B	1.293847644801005e-06	480
0.0625	TDRK5F	1.027672391629153e-10	641
	RK5W	1.144161054789095e-08	800
	RK5N	1.148518058435855e-08	960
	RK5B	3.918474900110880e-08	960
0.03125	TDRK5F	1.584399278442561e-12	1281
	RK5W	3.505891132959960e-10	1600
	RK5N	3.532993897437109e-10	1920
	RK5B	1.203353305889721e-09	1920
0.015625	TDRK5F	2.509104035652854e-14	2561
	RK5W	1.078148681443736e-11	3200
	RK5N	1.094990764727299e-11	3840
	RK5B	3.726075004095719e-11	3840
0.0078125	TDRK5F	1.221245327087672e-15	5121
	RK5W	3.316236174555343e-13	6400
	RK5N	3.415046023746982e-13	7680
	RK5B	1.159405904616051e-12	7680

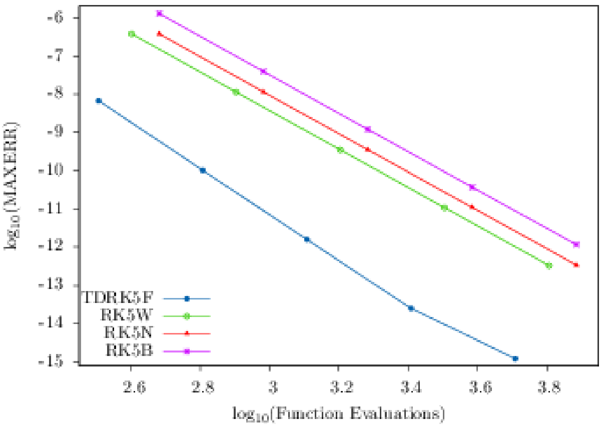

Figure 4. The performance curves with step size $h = 1/2^i, i = 3, 4, 5, 6, 7.$ for Problem 3.

Table 5. The Numerical Results for Problems 4

h	Method	MAXERR	FC
0.1	TDRK5F	2.385396100534898e-06	401
	RK5W	5.521056523183354e-06	500
	RK5N	6.441461286144090e-06	600
	RK5B	5.981424548195946e-05	600
0.05	TDRK5F	1.074797493227919e-07	801
	RK5W	3.355656120751505e-07	1000
	RK5N	2.026292326151591e-07	1200
	RK5B	1.888796089977163e-06	1200
0.025	TDRK5F	4.510416151681795e-09	1601
	RK5W	2.067595450405690e-08	2000
	RK5N	6.348464864913694e-09	2400
	RK5B	5.930724311653535e-08	2400
0.0125	TDRK5F	1.656299541963335e-10	3201
	RK5W	1.282923878243025e-09	4000
	RK5N	1.986670827847092e-10	4800
	RK5B	1.857541898075965e-09	4800
0.00625	TDRK5F	5.857536677922326e-12	6401
	RK5W	7.962897008439995e-11	8000
	RK5N	6.445732836368734e-12	9600
	RK5B	5.785072421105042e-11	9600

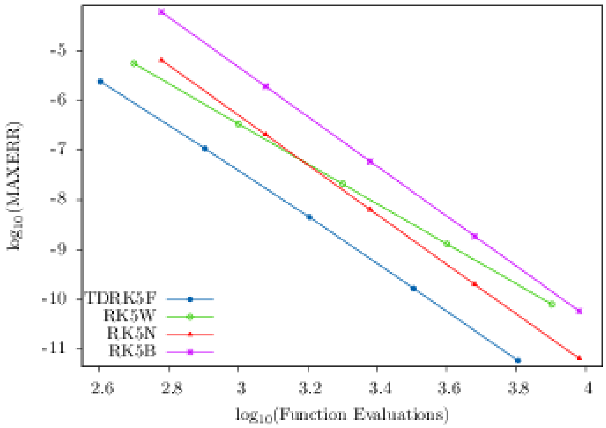

Figure 5. The performance curves with step size $h = 0.1/2^i, i = 0, 1, 2, 3, 4.$ for Problem 4.

Table 6. The Numerical Results for Problems 5

h	Method	MAXERR	FC
0.1	TDRK5F	2.295756667437399e-02	401
	RK5W	1.241930639620350e-01	500
	RK5N	1.241724140552514e-01	600
	RK5B	5.947253574392014e-01	600
0.05	TDRK5F	4.304830287424968e-04	801
	RK5W	1.068897298817984e-03	1000
	RK5N	1.068254081314168e-03	1200
	RK5B	4.739714669978024e-03	1200
0.025	TDRK5F	6.843461654172656e-06	1601
	RK5W	7.701960732076074e-06	2000
	RK5N	7.717269661755566e-06	2400
	RK5B	9.773522581246752e-06	2400
0.0125	TDRK5F	1.059042478157579e-07	3201
	RK5W	8.795918041426543e-07	4000
	RK5N	8.802996185330869e-07	4800
	RK5B	2.738380274536212e-06	4800
0.00625	TDRK5F	1.643343607027337e-09	6401
	RK5W	3.735857315168012e-08	8000
	RK5N	3.739636024457926e-08	9600
	RK5B	1.231891475494962e-07	9600

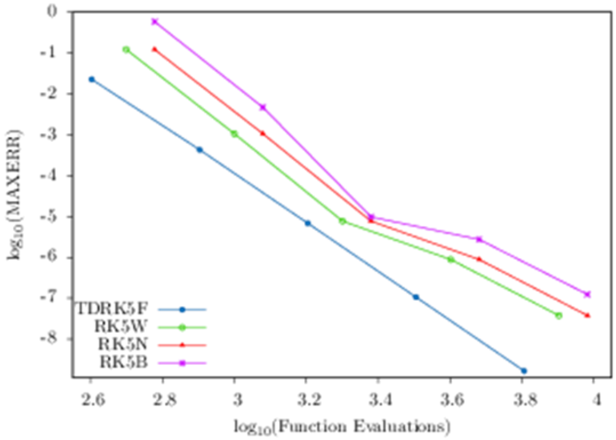

Figure 6. The performance curves with step size $h = 0.1/2^i, i = 0, 1, 2, 3, 4.$ for Problem 5.

It can be observed from Tables 2–6 and Figures 2–6 that the new TDRK5F method is more efficient than the Runge-Kutta methods chosen from the scientific literature in terms of accuracy and the number of function evaluations per each step.

Conclusion:

A new explicit two derivative Runge-Kutta method of order five with FSAL property is developed in this paper. Also, the linear stability of the new method is analyzed. From numerical results, we conclude that the new TDRK5F method is more efficient compared with the existing RK methods of the same order in the literature in terms of the number of function evaluations and the accuracy per step. The computations were implemented on a DELL PC with i3-3227U CPU, 4.0GB memory.

Acknowledgements:

The authors thank Mustansiriyah University, College of Science, Department of Mathematics for supported this work. We also express our gratitude to the referees for carefully reading the paper and for their valuable comments.

Conflicts of Interest: None.

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المشتقة الثانية الأول نفس الأخير طريقة رنك-كوتا الجديدة من الرتبة الخامسة مع اربعة مراحل

قاسم عباس حسين **زينب عيسى عبد النبي**

قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد، العراق

الخلاصة:

المشتقة الثانية طريقة رنك-كوتا الفعالة الجديدة من الرتبة الخامسة (TDRK) قد تم تطويرها من أجل الحل العددي للمعادلات التفاضلية الاعتيادية من الرتبة الأولى (ODEs). تم اشتقاق الطريقة الجديدة باستخدام خاصية الأول نفس الأخير (FSAL) . قمنا بتحليل استقرار الطريقة. تم عرض النتائج العددية لتوضيح كفاءة الطريقة الجديدة بالمقارنة مع بعض طرق رنك-كوتا (RK) المعروفة.

الكلمات المفتاحية: الطرق الصريحة ، خاصية الأول نفس الأخير FSAL ، مسائل القيم الأبتدائية ، المشتقة الثاتية طريقة رنك-كوتا.