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Study of T-norms and Quantum Logic Functions on BL-algebra and Their Relationships to the Classical Probability Structures

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Abstract

This paper is concerned with the study of the T-norms and the quantum logic functions on BL-algebra, respectively, along with their association with the classical probability space. The proposed constructions depend on demonstrating each type of the T-norms with respect to the basic probability of binary operation. On the other hand, we showed each quantum logic function with respect to some binary operations in probability space, such as intersection, union, and symmetric difference. Finally, we demonstrated the main results that explain the relationships among the T-norms and quantum logic functions. In order to show those relations and their related properties, different examples were built.

Keywords: BL-algebra, T-norms, Quantum Logic Functions, Probability Space, States.

دراسة الدوال المترية الاحتمالية ودوال المنطق الكمي على الفضاء الجبر الاساسي المنطقي وعلاقاتها بالنسبة الى الهياكل الاحتمالية التقليدية

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الخلاصة

هذا البحث يهتم بدراسة الدوال المترية الاحتمالية او ما تسمى بالدوال المعيارية، ودوال المنطق الكمي على الجبر الأساسي المنطقي بشكل متعاقب، وعلاقتها بفضاء الاحتمالية التقليدية. تعتمد البناءات المقترحة على اظهار كل نوع من أنواع الدوال المترية الاحتمالية بالنسبة لبعض العمليات الثنائية على فضاء الاحتمالية الاساسية للعملية الثنائية. من جهة أخرى وضحنا كل دالة منطق كمي بالنسبة لهذه العمليات الثنائية على الفضاء الاحتمالي مثل التقاطع، الاتحاد، والفرق. أخيرا، اظهرنا النتائج الرئيسية التي تبين العلاقات بين الدوال المترية الاحتمالية ودوال المنطق الكمي. لغرض إيضاح هذه العلاقات وخواصها المتعلقة بها بُنيت عدة امثلة مختلفة.

Introduction

Basic Logic (BL) was introduced by Hájek in 1990 to construct an algebra proof of the theory of basic logic's completeness which has been taken a place in the continuous t-norms and the fuzzy logic topic [1].

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BL-algebras are certain types of residual lattices [2] and they were examined in several papers by Turunen [3, 4, 5, 6]. Indeed, various kinds of algebras were examined among these algebras, for example, MV-algebras, G-algebras and BCK-algebras [1]. The structure $(A, \leq, \odot, \rightarrow, \vee, \wedge, O, I)$ of BL-algebra is said to be an MV-algebra if the complement operation $*: A \rightarrow A$ is involutive, which means that $\psi^{**} = \psi$ or equivalently $(\psi \rightarrow \lambda) \rightarrow \lambda = (\lambda \rightarrow \psi) \rightarrow \psi$ for all $\psi, \lambda \epsilon A$ where $\psi^* = \psi \rightarrow O$ [1]. Also, a BL-algebra is called a G-algebra if $\psi \odot \psi = \psi$ (i.e idempotent). A Boolean algebra is a BL-algebra which is both an MV-algebra and a G-algebra [4].

On other hand, triangular norms are the operations that looked to be suitable as well as possible to the notion of conjunction. When continuity is also required to be connectives, then the common part of all possible that have many-valued logics has been defined and called basic logic [7, 8].

Triangular norms started by Menger's paper "Statistical metrics" [9]. The first idea was to study metric spaces where probability distributions rather than numbers are used to model the distance between elements of the space in question. Triangular norms are derived into the form in the path of generalization of the classical triangle inequality which is the interesting condition in "metric space". There are four types of T-norms , namely the drastic product, the minimum, the product, and the Lukasiewicz.

Thus, the top field where T-norms play a remarkable role was the theory of probabilistic metric spaces (or statistical metric spaces as called after 1964). Schweizer and Sklar [10] redefined and developed statistical metric spaces.

Triangular norms (for short T-norms) are an important system for version of the conjunction in fuzzy logics and for the intersection of fuzzy sets [1, 11]. It is important to know that the left continuity of the T-norm corresponds to BL-algebras, and for more interesting details, one can refer to previous articles [7, 12] which described the relationship between continuous T-norms and Boolean algebra.

There are many quantum logic functions that were defined on quantum structure, for example, the quantum logic functions that were presented by Nánásiová in (2003). These maps, such as s-map and q-map, play a major role in this study and in the finding of the appropriate ways of association of T-norms formulas and the quantum logic functions. An essential notion that has a main part in our constructions is known as the state, the definition and properties of which are equivalent to those of the probability space.

Essentially, this study is organized as follows: section two deals with some basic concepts of T-norms, BL-algebra, state, probability space, quantum logics functions, and their properties. Section three is devoted to demonstrate the most proposed notions of generalization of t-norms and T-conorms on BL-algebra, their relationship to probability space, and their relationship to quantum logic functions. Finally, some conclusions and future works are presented.

2 Basic concepts

There are several basic concepts that need to be presented in this part. It involves definitions and properties that represent the foundations of our constructions on BL-algebra.

We firstly start with basic T-norm definitions.

Definition 2.1 [12]. A binary operation T on the unit interval [0,1] such that $T: [0,1]^2 \rightarrow [0,1]$ is said to be a triangular norm (T-norm) where 1 is an identity element and T satisfies the following conditions for each ψ, λ and $\gamma \epsilon [0,1]$:

1. $T(\psi, \lambda) = T(\lambda, \psi)$;
2. $T(\psi, T(\lambda, \gamma)) = T(T(\psi, \lambda), \gamma)$;
3. $T(\psi, \lambda) \leq T(\psi, \lambda)$ whenever $\lambda \leq \gamma$;
4. $T(\psi, 1) = \psi$.

Definition 2.2 [12]. A binary operation T^*on the unit interval [0,1] such that $T^*: [0,1]^2 \rightarrow [0,1]$ is said to be a triangular conorm (T-conorm) where 0 is an identity element (i.e). It is a function which has the same conditions (1-3) for all ψ, λ and $\gamma \epsilon [0,1]$ and satisfies that $T^*(\psi, 0) = \psi$.

Definition 2.3 [7, 9]. An algebra $(A, \leq, \odot, \rightarrow, \vee, \wedge, O, I)$ of type (2,2,2,2, O, I) is said to be BL-algebra if the following conditions hold:

1. $(A, \vee, \wedge, O, I)$ is a bounded lattice;
2. $(A, \odot, I)$ is a commutative monoid, such that $\odot$ is an associative and commutative binary operation, and I is a neutral element with respect to $\odot$;

3. $\psi \leq \lambda \to \gamma \leftrightarrow \lambda \odot \psi \leq \gamma$;

4. $\lambda \wedge \gamma = \lambda \odot (\lambda \to \gamma)$;
5. $(\lambda \to \gamma) \vee (\gamma \to \lambda) = I$.

For all ψ, λ and $\gamma \in A$ and consider $\lambda^* = \lambda \to O$

It is essential to show some common properties of BL-algebra. In each BL-algebras, the following relations hold [1, 3]:

1. $\psi \odot (\psi \to \lambda) \leq \lambda$;
2. $\psi \leq \lambda \quad iff \quad \psi \to \lambda = I$;
3. $\psi \vee \lambda = [(\psi \to \lambda) \to \lambda] \wedge [(\lambda \to \psi) \to \psi]$;
4. $I \to \psi = \psi, \psi \to \psi = I, \psi \leq \lambda \to \psi, \psi \to I = I$;
5. $\psi^* \odot \psi = O$;
6. $\psi \odot \lambda = O, iff \ \psi \leq \lambda^* \ and \ \psi \leq \lambda \ implies \ \lambda^* \leq \psi^*$;
7. $\psi \vee \lambda = I \ implies \ \psi \odot \lambda = \psi \wedge \lambda$;
8. $(\psi \to \lambda) \to (\psi \to \gamma) = (\psi \wedge \lambda) \to \gamma$;
9. $\psi \leq \psi^{**}, O^* = I, \ I^* = O$;
10. $(\psi \odot \lambda)^{**} = \psi^{**} \odot \lambda^{**}$;
11. $If \ \psi^{**} \leq \psi^{**} \to \psi$, then $\psi^{**} = \psi$.

Example 2.1 [24] Let $A = \{O, \psi, \lambda, I\}$ such that $\psi < \lambda$. Define on A the following operations explaining that A is a BL-algebra:

$\odot$	O	ψ	λ	I
O	O	O	O	O
ψ	O	O	ψ	ψ
λ	O	ψ	λ	λ
I	O	ψ	λ	I

$\to$	O	ψ	λ	I
O	I	I	I	I
ψ	ψ	I	I	I
λ	O	ψ	I	I
I	O	ψ	λ	I

$\wedge$	O	ψ	λ	I
O	O	O	O	O
ψ	O	ψ	ψ	ψ
λ	O	ψ	λ	λ
I	O	ψ	λ	I

Example 2.2. On BL-algebra $A = \{O, \psi, \psi^*, \lambda, \lambda^*, I\}$, define $\odot, \to, \wedge$ and $\vee$ as the following implies $\leq \lambda$:

$\odot$	O	ψ	ψ^*	λ	λ^*	I
O	O	O	O	O	O	O
ψ	O	ψ	O	ψ	λ^*	ψ
ψ^*	O	O	ψ^*	λ	λ^*	ψ^*
λ	O	ψ	λ	λ	O	λ
λ^*	O	λ^*	λ^*	O	λ^*	λ^*
I	O	ψ	ψ^*	λ	λ^*	I

$\to$	O	ψ	ψ^*	λ	λ^*	I
O	I	I	I	I	I	I
ψ	ψ^*	I	ψ^*	ψ^*	ψ^*	I
ψ^*	ψ	ψ	I	λ	ψ	I
λ	λ^*	ψ	ψ^*	I	λ^*	I
λ^*	λ	λ	ψ^*	λ	I	I
I	O	ψ	ψ^*	λ	λ^*	I

$\vee$	O	ψ	λ	I
O	O	ψ	λ	I
ψ	ψ	ψ	ψ	I
λ	λ	λ	λ	I
I	I	I	I	I

$\wedge$	O	ψ	ψ^*	λ	λ^*	I
O	O	O	O	O	O	O
ψ	O	ψ	O	O	O	ψ
ψ^*	O	O	ψ^*	λ	O	ψ^*
λ	O	ψ	λ	λ	O	λ
λ^*	O	O	λ^*	O	λ^*	λ^*
I	O	ψ	ψ^*	λ	λ^*	I

$\vee$	O	ψ	ψ^*	λ	λ^*	I
O	O	ψ	ψ^*	λ	λ^*	I
ψ	ψ	ψ	I	λ	ψ	I
ψ^*	ψ^*	I	ψ^*	I	ψ^*	I
λ	λ	λ	I	λ	I	I
λ^*	λ^*	ψ	ψ^*	I	λ^*	I
I	I	I	I	I	I	I

Definition 2.4. [13] Let $\mathcal{A}$ be a BL-algebra. Two elements $\psi, \lambda \in A$ are said to be orthogonal and denoted by $\psi \perp \lambda$, if $\psi^{**} \leq \lambda^{*}$.

Definition 2.5 [13]. Let $\mathcal{A}$ be a BL-algebra. A function $s: A \to [0,1]$ is said to be a state if the following conditions hold:

1.$s(O) = 0$

2. If $\psi \perp \lambda$, then s($\psi \vee \lambda$) = s(ψ) + s(λ).

Some properties of state:

1. $s(I) = 1$;

2. $s(\psi^*) = 1 - s(\psi)$ for any $\in A$;

3. s(ψ) = s(ψ^{**}) for any $\psi \in A$;

4. If $\psi \leq \lambda$, $s(\lambda)$ − s(ψ) = $1 - s(\psi \vee \lambda^*)$ which means that $1 - s(\psi \vee \lambda^*) = 1 - s(\psi^* \vee \lambda^*)$ **,** then $s(\psi) \leq s(\lambda)$.

Example 2.3. If $A = \{O, \psi, \lambda, I\}$ is a BL-algebra **Example 2.1**, then according to the table below a function $s: A \to [0,1]$ is a state.

.	O	ψ	λ	I
s(.)	0	0.1	0.9	1

Another definition that should be recalled is the definition of classical probability space.

Definition 2.6 [14, 15]. Let $(\Omega, \mathcal{F})$ be a measurable space. A map $\eta: \mathcal{F} \to [0,1]$ is said to be a measure if the following conditions hold:

(1) $\eta(H) \geq 0$ for all $H \in \mathcal{F}$;

(2) $\eta(\emptyset) = 0$;

(3) If $H_1, H_2, \ldots \in \mathcal{F}$, then $H_i \cap H_j = \emptyset$ for $i \neq j$ then

$\eta(\cup_{i=1}^{\infty} H_i) = \sum_{i=1}^{\infty} H_i$.

A measure ηis is said to be finite, if there is $k \in \mathcal{R}$ such that $\eta(\Omega) = k$.

Remark 2.1. It has been mentioned that probability space is homomorphism to BL-algebra [16]. This means taht the systems $(\Omega, \mathcal{F}, P, \cap, \cup, \emptyset)$ and $\mathcal{A} = (A, \leq, \odot, \to, \vee, \wedge)$ are homomorphism. Then $\cup \equiv \vee$, $\cap \equiv \odot \equiv \wedge$, $A^c \cup B \equiv a \to b$.

Definition 2.7 [14]. Let $(\Omega, \mathcal{F}, P)$be a probability space. The elements of σ-algebra $\mathcal{F}$ are said to be events, which are the set of outcomes of an experiment for which one can ask a probability.

It is a well-known fact that for all $H, G \in \mathcal{F}$:

$H = (H \cap G) \cup (H \cap G^c)$.

This property means that all events are simultaneously measurable in a probability space. In this case, we say that *H* and *G* are compatible.

3 T-norms and Quantum Logic Functions on BL-algebra

This part is devoted to demonstrate the constructions of T-norms and quantum logic functions on BL-algebras, respectively, and to show their relationships to probability space. We firstly begin with the definition of T-norm on BL-algebra.

Definition 3.1. Let $\mathcal{A}$ be a BL-algebra. A bivariate T-norm on BL-algebra, briefly s-T-norm, is a function $BT_S: A \times A \rightarrow [0,1]$ that fulfills the following conditions:
1. For each $\psi, \lambda \in$ A, $BT_S(\psi, O) = BT_S(O, \psi) = 0$;
2. For each $\psi, \lambda \in$ A, $BT_S(\psi, \lambda) = BT_S(\lambda, \psi)$;
3. For each $\psi, \lambda, \gamma \in$ A , $BT_S(\psi, \lambda) \leq BT_S(\psi, \gamma)$, if $\lambda \leq \gamma$;
4. $BT_S(., I)$ and $BT_S(I, .)$ are states.

Indeed, the generalization within this definition and its properties represent the essential brick that many of the properties and concepts are relevant to. The above definition can be modified to have a one more important property which is related to the notion of state.

Example 3.1. From **Example 2.1**, A is BL-algebra, then the following table satisfies s-T-norm conditions such that $BT_S(\psi, \lambda) = \min(s(\psi), s(\lambda))$ where s is a state.

$BT_S(.,.)$	O	ψ	λ	I
O	0	0	0	0
ψ	0	0.3	0.3	0.3
λ	0	0.3	0.5	0.5
I	0	0.3	0.5	1

Definition 3.2. Let $\mathcal{A}$ be a BL-algebra. A bivariate T-norm on BL-algebra, briefly s-T-norm, is a function $BT_S^*: A \times A \rightarrow [0,1]$ that fulfills the following conditions:
1. For each $\psi, \lambda \in A$, $BT_S^*(\psi, I) = BT_S^*(I, \psi) = 1$;
2. For each $\psi, \lambda \in A$, $BT_S^*(\psi, \lambda) = BT_S^*(\lambda, \psi)$;
3. For each $\psi, \lambda, \gamma \in A$, $BT_S^*(\psi, \lambda) \leq BT_S^*(\psi, \gamma)$, if $\lambda \leq \gamma$;
4. $BT_S^*(., O)$ and $BT_S^*(O, .)$ are states.

Example 3.2 From example (2.1), A is a BL-algebra, if $BT_S^*(\psi, \lambda) = \max(s(\psi), s(\lambda))$, where s is a state, so that BT_S^* satisfies s -T-conorm.

Solution
1. Let $\psi, \lambda \in A$, $BT_S^*(\psi, I) = \max(s(\psi), s(I)) = 1 = BT_S^*(I, \psi)$;
2. Let $\psi, \lambda \in$ A , $BT_S^*(\psi, \lambda) = \max(s(\psi), s(\lambda)) = \max(s(\lambda), s(\psi)) = BT_S^*(\lambda, \psi)$
3. If $\lambda \leq \gamma$ such that $\in A$, then $BT_S^*(\psi, \lambda) = \max(s(\psi), s(\lambda)) \leq \max(s(\psi), s(\gamma)) = BT_S^*(\psi, \gamma)$, since s is a monotone;
4. $BT_S^*(., O)$ and $BT_S^*(O, .)$ are states.

For $BT_S^*(., O)$ we have:
i) $BT_S^*(O, O) = \max(s(O), s(O)) = 0$
ii) $BT_S^*(O, I) = \max(s(O), s(I)) = 1$
iii) If $\psi \perp \lambda$ then $BT_S^*(\psi \vee \lambda, O) = BT_S^*(\psi, O) + BT_S^*(\lambda, O)$
$BT_S^*(\psi, O) = \max(s(\psi), s(O)) = s(\psi)$
$BT_S^*(\lambda, O) = \max(s(\lambda), s(O)) = s(\lambda)$
$BT_S^*(\psi \vee \lambda, O) = \max\big(s(\psi \vee \lambda), s(O)\big) = \max(s(\psi), s(O)) + \max\big(s(\lambda), s(O)\big)$
$= s(\psi) + s(\lambda) = BT_S^*(\psi, O) + BT_S^*(\lambda, O)$.
Similarly, we can obtain $BT_S^*(O, .)$.
Therefore $BT_S^*(\psi, \lambda) = \max(s(\psi), s(\lambda))$ is an s-T-conorm.

Remark 3.1. Note that the s-T norm and s-T-conorm cannot fulfill the associative condition of classical T-norms because of the difference between the domains and the range of these new functions. Indeed, we can leave this as an open problem to our next study. Nevertheless, it is interesting to focus on other properties of these constructions because they are rich and yield many modified constructions.

Now, turn to the notions of quantum logic functions and their structures on BL-algebra. Indeed, these functions and their concepts were previously discussed in detail [17, 18] and constructed on an orthomodular lattice. According to the notions demonstrated in the literatures above, the system of the orthomodular lattice and its related properties are a homomorphism to BL-algebra. In fact, this would help to reconstruct the quantum logic functions such as s-map, j-map, and d-map on BL-algebra.

Definition 3.3. Let $\mathcal{A}$ be a BL-algebra. A map p: A $\times$ A $\rightarrow$ [0,1] is called a bivariate s-map if the following conditions hold.
(S1) $p(I,I) = 1$;
(S2) For every $\psi, \lambda \in A$, if $\psi \perp \lambda$ then $p(\psi,\lambda) = 0$;
(S3) If $\psi \perp \lambda$ then for any $\gamma \in A$,
i. $p(\psi \vee \lambda, \gamma) = p(\psi,\gamma) + p(\lambda,\gamma)$;
ii. $p(\gamma, \psi \vee \lambda) = p(\gamma,\psi) + p(\gamma,\lambda)$.

Example 3.3. Let $A = \{O, \psi, \psi^*, \lambda, \lambda^*, I\}$ be a BL-algebra **Example 2.2**. Then the table below shows that p is a bivariate s-map.

$p(.,.)$	O	ψ	ψ^*	λ	λ^*	I
O	0	0	0	0	0	0
ψ	0	0.3	0	0.2	0.1	0.3
ψ^*	0	0	0.7	0.3	0.4	0.7
λ	0	0.12	0.38	0.5	0	0.5
λ^*	0	0.18	0.32	0	0.5	0.5
I	0	0.3	0.7	0.5	0.5	1

Definition 3.4. Let $\mathcal{A}$ be a BL-algebra. A join map (for short j-map) is a map q: A $\times$ A $\rightarrow$ [0,1] such that the following conditions hold:
(q1) $q(O,O) = 0$ *and* $q(I,I) = 1$;
(q2) For each $\psi, \lambda \in A$, if $\psi \perp \lambda$ then $q(\psi,\lambda) = q(\psi,\psi) + q(\lambda,\lambda)$;
(q3) If $\psi \perp \lambda$ then for each $\in A$,
$q(\psi \vee \lambda, \gamma) = q(\psi,\gamma) + q(\lambda,\gamma) - q(\gamma,\gamma)$
$q(\gamma, \psi \vee \lambda) = q(\gamma,\psi) + q(\gamma,\lambda) - q(\gamma,\gamma)$.

Example 3.4. Let $A = \{O, \psi, \psi^*, \lambda, \lambda^*, I\}$ be a BL-algebra **Example 2.2**. Then the table below shows that q is a bivariate j-map.

$q(.,.)$	O	ψ	ψ^*	λ	λ^*	I
O	0	0.7	0.3	0.67	0.33	1
ψ	0.7	0.7	1	0.9	0.8	1
ψ^*	0.3	1	0.3	0.77	0.53	1
λ	0.67	0.88	0.79	0.67	1	1
λ^*	0.33	0.82	0.51	1	0.33	1
I	1	1	1	1	1	1

Definition 3.5. Let $\mathcal{A}$ be a BL-algebra. A difference map (d-map) is a map d: A $\times$ A $\rightarrow$ [0,1] such that the following conditions hold:
(d1) $d(I,O) = d(O,I) = 1$, for all $\psi \in A, d(\psi,\psi) = 0$;
(d2) $d(\psi,\lambda) = d(\psi,O) + d(O,\lambda)$ whenever $\perp$ λ ;
(d3) If $\psi \perp \lambda$ and $\gamma \in A$,
$d(\psi \vee \lambda, \gamma) = d(\psi,\gamma) + d(\lambda,\gamma) - d(O,\gamma)$
$d(\gamma, \psi \vee \lambda) = d(\gamma,\psi) + d(\gamma,\lambda) - d(\gamma,O)$.

Example 3.5. Let $A = \{O, \psi, \psi^*, \lambda, \lambda^*, I\}$ be a BL-algebra **Example 2.2**. Then the table below shows that d is a bivariate d-map.

$d(.,.)$	O	ψ	ψ^*	λ	λ^*	I
O	0	0.48	0.52	0.86	0.14	1
ψ	0.48	0	1	0.13	0.87	0.52
ψ^*	0.52	1	0	0.87	0.13	0.48
λ	0.86	0.3	0.7	0	1	0.14
λ^*	0.14	0.7	0.3	1	0	0.86
I	1	0.52	0.48	0.14	0.86	0

Now, it is convenient to build the constructions that connect BT_S and BT_S^* to BL-algebra functions (s-map and j-map) through probability space. Note that each element a belongs to $\mathcal{A}$ is equivalent to each event that belong to $\mathcal{F}$.

Theorem 3.1. Let BT_S be an s-T-norm on a BL-algebra $\mathcal{A}$ with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then:

$$BT_S(H, G) = P(H \cap G)$$

Proof

The proof should show that the relation above satisfies the conditions of BT_S function.

1. $BT_S(H, \emptyset) = P(H \cap \emptyset) = P(\emptyset) = 0$;
2. $BT_S(H, G) = P(H \cap G) = P(G \cap H) = BT_S(G, H)$ ($\cap$ is commutative);
3. Let $G \subseteq C$. Then, $BT_S(H, G) = P(H \cap G)$, and $BT_S(H, C) = P(H \cap C)$, but $P(\mathrm{H} \cap G) \leq P(H \cap C)$, ($\cap$ is a monotone). Therefore, $BT_S(H, G) \leq BT_S(H, C)$;
4. To prove that $BT_S(\Omega, .)$ and $BT_S(., \Omega)$ are states, we need to prove the following.

For $BT_S(., \Omega)$, we have:

i) $BT_S(\Omega, \Omega) = P(\Omega \cap \Omega) = 1$;

ii) $BT_S(\emptyset, \Omega) = P(\emptyset \cap \Omega)) = P(\emptyset) = 0$;

iii) Let $H, G \in A$ such that $H \cap G = \emptyset$, then $BT_S(H \cup G, \Omega) = BT_S(H, \Omega) + BT_S(G, \Omega)$

$BT_S(H \cup G, \Omega) = \mathrm{P}((\mathrm{H} \cup \mathrm{G}) \cap \Omega) = \mathrm{P}(\mathrm{H} \cap \Omega) + \mathrm{P}(\mathrm{G} \cap \Omega) = BT_S(\mathrm{H}, \Omega) + BT_S(\mathrm{G}, \Omega)$

Hence, $BT_S(., \Omega)$ is a state

Similarly, $BT_S(\Omega, .)$ is a state too.

Therefore, $BT_S(H, G) = P(H \cap G)$ is an s-T-norm.

Theorem 3.2. Let BT_S^* be an s-T-conorm on a BL-algebra $\mathcal{A}$ with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then

$$BT_S^*(\mathrm{H}, \mathrm{G}) = \mathrm{P}(\mathrm{H} \cup \mathrm{G}).$$

Proof

The proof should show that the relation above satisfies the conditions of BT_S function.

1. $BT_S^*(\mathrm{H}, \emptyset) = \mathrm{P}(\mathrm{H} \cup \Omega) = \mathrm{P}(\Omega) = 1$;
2. $BT_S^*(\mathrm{H}, \mathrm{G}) = \mathrm{P}(\mathrm{H} \cup \mathrm{G}) = \mathrm{P}(\mathrm{G} \cup \mathrm{H}) = BT_S^*(\mathrm{G}, \mathrm{H})$ ($\cup$ is commutative);
3. Let $G \subseteq C$. Then , $BT_S^*(\mathrm{H}, \mathrm{G}) = \mathrm{P}(\mathrm{H} \cup \mathrm{G})$ and $BT_S^*(\mathrm{H}, \mathrm{C}) = \mathrm{P}(\mathrm{H} \cup \mathrm{C})$, but $\mathrm{P}(\mathrm{H} \cup \mathrm{G}) \leq \mathrm{P}(\mathrm{H} \cup \mathrm{C})$ ($\cup$ is a monotone). Therefore, $BT_S^*(\mathrm{H},\mathrm{G}) \leq BT_S^*(\mathrm{H},\mathrm{C})$;
4. To prove that $BT_S^*(\emptyset,.)$ and $BT_S^*(., \emptyset)$ are states, we need to prove the following.

For $BT_S^*(., \emptyset)$ we have:

i) $BT_S^*(\emptyset, \emptyset) = \mathrm{P}(\emptyset \cup \emptyset) = 0$;

ii) $BT_S^*(\Omega, \emptyset) = \mathrm{P}(\Omega \cup \emptyset)) = \mathrm{P}(\Omega) = 1$;

iii) Let H,G $\in A$ such that H∩G= ∅, then $BT_S^*(\mathrm{H} \cup \mathrm{G}, \emptyset) = BT_S^*(\mathrm{H}, \emptyset) + BT_S^*(\mathrm{G}, \emptyset)$

$BT_S^*(\mathrm{H} \cup \mathrm{G}, \emptyset) = \mathrm{P}((\mathrm{H} \cup \mathrm{G}) \cup \emptyset) = \mathrm{P}(\mathrm{H} \cup \emptyset) + \mathrm{P}(\mathrm{G} \cup \emptyset) = BT_S^*(\mathrm{H},\emptyset) + BT_S^*(\mathrm{G},\emptyset)$

Hence, $BT_S^*(., \emptyset)$ is a state

Similarly, $BT_S^*(\emptyset,.)$ is a state too.

Therefore, $BT_S^*(H, G) = P(H \cup G)$ is an s-T-conorm.

On the other hand, s-map, j-map, and d-map can be modified in terms of probability space as follow:

Theorem 3.3. Let p be an s-map on a BL-algebra with a probability space $(\Omega, \mathcal{F}, P)$.Then

$$p(H, G) = P(H \cap G).$$

Proof

It has to be shown that the conditions of the s-map are satisfied within the above relation.

That is:

1. $p(\Omega, \Omega) = P(\Omega \cap \Omega) = 1$;
2. For every $H, G \in \mathrm{A}$, if H ∩ G =∅, then $p(\mathrm{H}, \mathrm{G}) = P(\mathrm{H} \cap \mathrm{G}) = P(\emptyset) = 0 = p(G, H)$;
3. If H ∩ G = ∅, and C ∈ A. Then

$$p(\mathrm{H} \cup \mathrm{G}, \mathrm{C}) = P\big((\mathrm{H} \cup \mathrm{G}) \cap \mathrm{C}\big) = P((\mathrm{H} \cap \mathrm{C}) \cup (\mathrm{G} \cap \mathrm{C}))$$
$$= P((\mathrm{H} \cap \mathrm{C}) + \mathrm{P}(\mathrm{G} \cap \mathrm{C}) = p(H, \mathrm{C}) + p(\mathrm{G}, \mathrm{C}).$$

Similarly, $\boldsymbol{p}(\mathbf{C}, \mathbf{H} \cup \mathbf{G}) = \boldsymbol{p}(\mathbf{C}, \mathbf{H}) + \boldsymbol{p}(\mathbf{C}, \mathbf{G})$.

Theorem 3.4. Let q be an j-map on a BL-algebra with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then $q(\mathrm{H}, \mathrm{G}) = P(\mathrm{H} \cup \mathrm{G})$.

Proof

Again, it is essential to show that the conditions of the j-map hold:

1. $q(\emptyset, \emptyset) = P(\emptyset \cup \emptyset) = P(\emptyset) = 0 \;\; and \;\; q(\Omega, \Omega) = P(\Omega \cup \Omega) = P(\Omega) = 1$;
2. For every $\mathrm{H}, \mathrm{G} \in \mathrm{A}$, if $\mathrm{H} \cap \mathrm{G} = \emptyset$, then $p(\mathrm{H}, \mathrm{G}) = P(\mathrm{H} \cup \mathrm{G}) = P(H \cup H) + P(G \cup G)$;
3. Let $\mathrm{H}, \mathrm{G} \in A$, such that $\mathrm{H} \cap \mathrm{G} = \emptyset$. Then for each $\mathrm{C} \in A$

$q(\mathrm{H} \cup \mathrm{G}, \mathrm{C}) = P\big((\mathrm{H} \cup \mathrm{G}) \cup \mathrm{C}\big) = P((\mathrm{H} \cup \mathrm{G}) \cup (\mathrm{C} \cup \mathrm{C}))$

$P(H \cup C) + P(G \cup C) - P\big((H \cup C) \cap (G \cup C)\big) = P(H \cup C) + P(G \cup C) - P((H \cap G) \cup C)$

$= \mathbf{P}(\mathbf{H} \cup \mathbf{C}) + \mathbf{P}(\mathbf{G} \cup \mathbf{C}) - \mathbf{P}(\emptyset \cup \mathbf{C})$

but $\boldsymbol{C} = \boldsymbol{C} \cup \boldsymbol{C}$

Hence, $\boldsymbol{q}(\boldsymbol{H} \cup \boldsymbol{G}, \boldsymbol{C}) = \boldsymbol{P}(\boldsymbol{H} \cup \boldsymbol{C}) + \boldsymbol{P}(\boldsymbol{G} \cup \boldsymbol{C}) - \boldsymbol{P}(\boldsymbol{C} \cup \boldsymbol{C})$

therefore, $\boldsymbol{q}(\boldsymbol{H} \cup \boldsymbol{G}, \boldsymbol{C}) = \boldsymbol{q}(\boldsymbol{H}, \boldsymbol{C}) + \boldsymbol{q}(\boldsymbol{G}, \boldsymbol{C}) - \boldsymbol{q}(\boldsymbol{C}, \boldsymbol{C})$.

Similarly, $\boldsymbol{q}(\boldsymbol{C}, \boldsymbol{H} \cup \boldsymbol{G}) = \boldsymbol{q}(\boldsymbol{C}, \boldsymbol{H}) + \boldsymbol{q}(\boldsymbol{C}, \boldsymbol{G}) - \boldsymbol{q}(\boldsymbol{C}, \boldsymbol{C})$.

Also, the relationship between the d-map and the the probability space of difference could be shown as follows:

Theorem 3.5. Let d be a d-map on a BL-algebra with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then

$$d(H, G) = P(H \Delta G).$$

Proof

It is essential to show that the conditions of the d-map hold:

1. $d(\emptyset, \Omega) = P(\emptyset \Delta \Omega) = P(\Omega \Delta \emptyset = d(\Omega \Delta \emptyset) = P(\Omega) = 1$, where for each $\mathrm{H} \in A$, $d(\mathrm{H}, \mathrm{H}) = P(H \Delta H) = 0$;
2. Let $H, G \in A$, such that $H \cap G = \emptyset$. $d(H, G) = P(H \Delta G) = P[(H^c \cap G) \cup (H \cap G^c)]$

$= P[((H^c \cap G) \cup \mathrm{H}) \cap ((H^c \cap G) \cup G^c)]$

$= P[((H^c \cup \mathrm{H}) \cap (G \cup H)) \cap ((H^c \cup G^c) \cap (\mathrm{G} \cup G^c))]$

$= P[(\Omega \cap (G \cup H)) \cap ((H^c \cup G^c) \cap \Omega)]$, but $H^c \cup G^c = (H \cap G)^c$

Thus

$= P[G \cup H) \cap (H \cap G)^c] = P[(G \cup H) \cap \emptyset^c] = P[(G \cup H) \cap \Omega] = P(G \cup H) = P(G) + P(H)$

$= P(H \Delta \emptyset) + P(\emptyset \Delta G) = d(H, \emptyset) + d(\emptyset, G)$;

3. Let $\mathrm{H}, \mathrm{G} \in A$, such that $H \cap G = \emptyset$. Then for each $C \in A$,

$d(\mathrm{H} \cup \mathrm{G}, \mathrm{C}) = P\big((\mathrm{H} \cup \mathrm{G}) \Delta \mathrm{C}\big) = P((\mathrm{H} \Delta \mathrm{C}) \cup (\mathrm{G} \Delta \mathrm{C}))$

$= \mathbf{P}(\mathbf{H} \Delta \mathbf{C}) + \mathbf{P}(\mathbf{G} \Delta \mathbf{C}) - \mathbf{P}((\mathbf{H} \Delta \mathbf{C}) \cap (\mathbf{G} \Delta \mathbf{C}))$

$= \mathbf{P}(\mathbf{H} \Delta \mathbf{C}) + \mathbf{P}(\mathbf{G} \Delta \mathbf{C}) - \mathbf{P}((\mathbf{H} \cap \mathbf{G}) \Delta \mathbf{C})$

$= \mathbf{P}(\mathbf{H} \Delta \mathbf{C}) + \mathbf{P}(\mathbf{G} \Delta \mathbf{C}) - \mathbf{P}(\emptyset \Delta \mathbf{C})$

$= \boldsymbol{d}(\mathbf{H}, \mathbf{C}) + \boldsymbol{d}(\mathbf{G}, \mathbf{C}) - \boldsymbol{d}(\emptyset, \mathbf{C})$.

Similarly, $\boldsymbol{d}(\mathbf{C}, \mathbf{H} \cup \mathbf{G}) = \mathbf{P}\big(\mathbf{C} \Delta (\mathbf{H} \cup \mathbf{G})\big) = \boldsymbol{d}(\mathbf{C}, \boldsymbol{H}) + \boldsymbol{d}(\mathbf{C}, \boldsymbol{G}) - \boldsymbol{d}(\mathbf{C}, \emptyset)$.

This completes the proof.

In fact, the theorems above directly lead us to the following results.

Corollary 3.1. If $\mathbf{BT_S}$ be an s-T-norm, and $\boldsymbol{p}$ be an s-map on a BL-algebra , respectively, then each s-map $\boldsymbol{p}$ is $\mathbf{BT_S}$.

Proof

It is easy to see that the proof directly follows from **Theorem (3.1)** and **Theorem (3.3)**. Then th proof is complete.

Corollary 3.2. If $\boldsymbol{BT_S^*}$ be an s-T-conorm, and $\boldsymbol{q}$ be a j-map on a BL-algebra $\boldsymbol{\mathcal{A}}$, respectively, then each j-map $\boldsymbol{q}$ is $\boldsymbol{BT_S^*}$.

Proof

From **Theorem 3.2** and **Theorem 3.4,** the prove is complete.

Remark 3.2. According to the notions above (**Corollary 3.1**, and **Corollary 3.2**), it is important to know that each s-map and j-map are s-T-norms s-T-conorms, respectively. But the converse is not true and each type of s-T-norm is not need to fulfill the conditions of quantum logic functions (s-map, j-map).

4 Conclusions

There are several properties that can be summarized in the following sentences. First of all, it is clear that T-norms, and the quantum logic functions on BL-algebra, have much complicated structures than the classical ones because of the nature of BL-algebra. As a type of a generalization of T-norm and T-conorm, we showed several different properties that associate different situations of T-norm and quantum logic functions to the classical probability space. Also, it is essential to refer to the role of the state in our constructions and how it is useful in each case of generalization. Moreover, the proofs of many properties and facts associated with the calculations that have been obtained in the tables show different properties of each map on BL-algebra. Indeed, there are some open problems that we are working on, such as generalizing these concepts in the case of conditional events or independent events. Finally, it is good to investigate the associative conditions of T-norms and T-conorms on BL-algebra.

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