FACULTY OF SCIENCE

MASTER PROGRAM OF MATHEMATICS

The Convex Darboux Theorem and Applications

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Declaration

I certify that this Thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

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Abstract

In this thesis, we review the convex Darboux theorem by Ekeland and Nirenberg [14]. Moreover, we give the necessary and sufficient conditions for a smooth k-homogeneous differential 1-form ω defined in a neighborhood $\mathcal{U}$ of some point $\bar{x} \in \mathbb{R}^n$ to be decomposed as

$$\omega = f^1(x)dg_1(x) + \ldots + f^k(x)dg_k(x)$$

for all x in some neighborhood $\mathcal{V} \subset \mathcal{U}$ of $\bar{x}$ where $f^1, \ldots, f^k$ are homogeneous functions of arbitrary degree and $g_1, \ldots, g_k$ are homogeneous of degree zero. Finally, we give some economic applications to both results from consumer theory.

إهداء

إلى أمي وأبي اللذين يمتلئ بهم قلبي..

إلى معاذ صديقي الأقرب إليّ..

إلى كلّ مَن ترك بي أثراً طيباً يوماً..

إلى مشرفي في العمل الأستاذ مروان العقيلي..

الملخص

تهدف هذه الرسالة لدراسة نظرية داربو المقعرة و تطبيقاتها لكل من ايكلاند و نيرينبيرغ. بالإضافة لذلك سوف نعطي الشروط الضرورية والكافية لكتابة شكل تفاضلي متجانس على الشكل التالي

$$\omega = f^1 dg_1 + \cdots + f^k dg_k$$

في جوار نقطة معينة بحيث تكون الاقترانات $f^1, \ldots, f^k$ متجانسة من أي درجة والاقترانات $g_1, \ldots, g_k$ متجانسة من درجة الصفر. في النهاية، سوف نعطي بعض التطبيقات الاقتصادية للنتيجتين السابقتين من نظرية المستهلك.

Contents

Chapter 1

Some Basic Definitions and Results

We review some definitions and results that we will use in this thesis. A detailed exposition on the following definitions and the proof of next theorem can be found in [20].

Definition 1.1. [**Cone**] A cone C in $\mathbb{R}^n$ is a set of points such that if $x \in C$, then so is every positive scalar multiple of x, i.e, if $x \in C$, then $\lambda x \in C$ for all $\lambda > 0$.

Definition 1.2. [**Homogeneous Function**] Let $g : C \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function defined on a cone C. Then, g is said to be homogenous of degree $k \in \mathbb{R}$ (k-homogeneous) if for any real number $t > 0$, the following holds

$$g(tx^1, tx^2, \ldots, tx^n) = t^k g(x^1, x^2, \ldots, x^n), \quad \forall x \in C.$$

Theorem 1.1. ***[Euler's Theorem]*** *A C^1 function g is k-homogeneous on a cone $C \subset \mathbb{R}^n$ if and only if*

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x^i}(x) x^i = k g(x), \quad \forall x \in C.$$

Definition 1.3. **[Convex Set]** A set U is called convex if for any points x, y in U, the line segment joining x and y

$$l(x, y) = \{tx + (1-t)y : 0 \leq t \leq 1\}$$

is also in U.

Definition 1.4. **[Convex Function]** A real-valued function g defined on a convex set $U \subset \mathbb{R}^n$ is convex if for all x, y in U and $t \in [0, 1]$,

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y).$$

Definition 1.5. **[Strongly Convex Function]** A real-valued function $g(x)$ defined on a convex set $U \subset \mathbb{R}^n$ is strongly convex if there exists $\alpha > 0$ such that $g(x) - \frac{\alpha}{2}||x||^2$ is convex for all $x \in U$.

Definition 1.6. **[Quasiconvex Function]** A real-valued function g defined on a convex set $U \subset \mathbb{R}^n$ is quasiconvex if for all x, y in U and $t \in [0, 1]$,

$$g(tx + (1-t)y) \leq \max\{g(x), g(y)\}.$$

Definition 1.7. **[Positive Definite Matrices]** A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive definite if

$$x^T Ax > 0, \quad \forall x \neq 0 \in \mathbb{R}^n.$$

Definition 1.8. **[Positive Semidefinite Matrices]** A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive semidefinite if $x^T Ax \geq 0$ for all $x \neq 0 \in \mathbb{R}^n$.

Theorem 1.2. *Let U be an open convex set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a C^2 function. Then, f is a convex function on U if and only if $D^2 f(x)$ is a positive semidefinite matrix for all $x \in U$, where $D^2 f(x)$ is the Hessian* *matrix of $f(x)$ defined as*

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Theorem 1.3. *Let U be an open convex set of $\mathbb{R}^n$, and let $g : U \to \mathbb{R}$ be a C^2 function. Then, $g(x)$ is a strongly quasi-convex function on U if $D^2 g(x)$ is a positive definite matrix on $\{\nabla g(x)\}^{\perp}$.*

Theorem 1.4 (Envelope Theorem for Constrained Problems). *Let $x^*(a) \in \mathbb{R}^n$ denote the solution of the following problem:*

$$\max f(x; a)$$

$$s.t \quad g_1(x; a) = 0, \ldots, g_k(x; a) = 0.$$

Let $\lambda_1(a), \ldots, \lambda_k(a)$ be the Lagrange multipliers for each constraint in this problem. Then

$$\underbrace{\frac{d}{da} f(x^*(a), a)}_{\text{Total derivative for the original function } f} = \underbrace{\frac{\partial}{\partial a} L(x^*(a), \lambda(a), a)}_{\text{Partial derivative of Lagrange}}$$

Chapter 2

Exterior Differential Calculus

2.1 Differential Manifolds

Here we will define a differential manifold

Definition 2.1. [17][**Manifold**] A manifold M of dimension n is a topoloy-ical space M, such that each point $x \in M$ has a neighborhood which is homeomorphic to an open set in the Euclidean space $\mathbb{R}^n$.

Definition 2.2. [17][**Chart**] A chart for a manifold M is a subset U of M together with a bijective map

$$\phi : U \to \phi(U).$$

where $\phi(U) \subset \mathbb{R}^n$. Usually we denote the coordinates of a point $m \in U \subset M$ by $\phi(m) = (x^1, x^2, \ldots, x^n)$.

Definition 2.3. [17][**Compatible**] Two charts on a manifold M, (U, ϕ) and (U', ϕ') are called compatible, if $U \cap U' = \emptyset$, or $\phi(U \cap U')$ and $\phi'(U \cap U')$ are open subsets of $\mathbb{R}^n$ and the maps

$$\phi \circ (\phi')^{-1} : \phi'(U \cap U') \to \phi(U \cap U')$$

$$\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$$

are smooth.

Definition 2.4. [17][**Atlas**] A collection of charts

$$\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha | \alpha \in I\}$$

is called an atlas if for any pair of indices $i, j, (U_{\alpha_i}, \phi_{\alpha_i})$ and $(U_{\alpha_j}, \phi_{\alpha_j})$ are compatible and $\bigcup_{\alpha \in I} U_\alpha = M$.

Example 2.1. [17] The unit sphere

$$S^n = \{(a^1, a^2, \dots, a^{n+1}) \in R^{n+1} | (a^1)^2 + (a^2)^2 + \dots + (a^{n+1})^2 = 1\}$$

$(n \geq 1)$ has an atlas consisting of two charts, we construct as follow:
Any point $(a^1, a^2, \dots, a^{n+1}) \in S^n$, different from $(0, \dots, 0, 1)$, can be joined with $(0, \dots, 0, 1)$ by straight line that intersects the hyperplane $a^{n+1} = 0$ at some point $(b^1, b^2, \dots, b^n, 0)$. The condition that three points $(a^1, a^2, \dots, a^{n+1})$, $(0, \dots, 0, 1)$ and $(b^1, b^2, \dots, b^n, 0)$ lie on a straight line yields

$$(b^1, b^2, \dots, b^n, 0) - (0, \dots, 0, 1) = \lambda[(a^1, a^2, \dots, a^{n+1}) - (0, \dots, 0, 1)] \quad (2.1)$$

for some $\lambda \in R$. We consider the last component in the vector equation (2.1), we have

$$\lambda = \frac{1}{1 - a^{n+1}}$$

Substitute the value of λ in equation (2.1), we find a map

$$\phi : S^n \backslash (0, \dots, 0, 1) \to R^n$$

defined by

$$\phi(a^1, a^2, \dots, a^{n+1}) = \frac{1}{1 - a^{n+1}} (a^1, a^2, \dots, a^n)$$

The pair (U, ϕ), where $U = S^n \backslash (0, \ldots, 0, 1)$, is a chart for S^n, since ϕ is injective and $\phi(U) = R^n$.

In a similar manner, joining the points of S^n with $(0, \ldots, 0, -1)$, we obtain a map

$$\chi : S^n \backslash (0, \ldots, 0, -1) \to R^n$$

that given by

$$\chi(a^1, a^2, \ldots, a^{n+1}) = \frac{1}{1 + a^{n+1}}(a^1, a^2, \ldots, a^n).$$

The pair (V, χ), where $V = S^n \backslash (0, \ldots, 0, -1)$, is a chart for S^n.
Then, the unit sphere has an atlas consisiting of two charts (U, ϕ) and (V, χ).

Definition 2.5. [17] Two atlases are called equivalent if their union is also an atlas.

Definition 2.6. [17][**Differential Manifold**] A differential manifold is a set of points together with a finite set of subsets $U_i \subset M$ and one-to-one mappings

$$\phi_i : U_i \to R^n$$

such that

1. $M = \bigcup\limits_i U_i$.
2. For any nonempity intersection $U_i \cap U_j$, the set $\phi_i(U_i \cap U_j)$ is an open subset of R^n, and the one-to-one mapping $\phi_j \circ \phi_i^{-1}$ is a smooth function on $\phi_i(U_i \cap U_j)$.

Definition 2.7. [17] A differential manifold M is called an n-manifold if every chart has domain in an n-dimensional vector space.

2.2 Tangent Space

Two curves $t \to c_1(t)$ and $t \to c_2(t)$ in an n-manifold M are called equivalent at m if

$$c_1(0) = c_2(0) = m, \qquad (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$$

for some chart ϕ.

Remark 2.1. The Equivalence does not depend on the choice of chart.

Proof. Let $c_1(t)$ and $c_2(t)$ be equivalent curves in an n-manifold M at m, then

$$c_1(0) = c_2(0) = m, \qquad (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$$

for some chart ϕ.

If we change to a chart η, then

$$(\eta \circ c_1)'(0) = ((\eta \circ \phi^{-1}) \circ (\phi \circ c_1))'(0) = (\eta \circ \phi^{-1})'(\phi \circ c_1)'(0)$$

$$(\eta \circ c_2)'(0) = ((\eta \circ \phi^{-1}) \circ (\phi \circ c_2))'(0) = (\eta \circ \phi^{-1})'(\phi \circ c_2)'(0)$$

But, $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$, then

$$(\eta \circ c_1)'(0) = (\eta \circ c_2)'(0)$$

□

Definition 2.8. Let C be a differentiable curve in M and $f \in C^\infty(M)$, then $C^*f = f \circ C$ is a differentiable function from an open subset $I \subset R$ into R. If $t_0 \in I$, then the tangent vector to C at the point $C(t_0)$, denoted by C'_{t_0}, defined by

$$C'_{t_0}[f] = \frac{d}{dt}(C^*f)\bigg|_{t_0} = \lim_{t \to t_0} \frac{f(C(t)) - f(C(t_0))}{t - t_0}$$

Hence, C'_{t_0} is a map from $f \in C^\infty(M)$ into R with the properties

i. $C'_{t_0}[af+bg] = aC'_{t_0}[f]+bC'_{t_0}[g]$, for all $a, b \in R$ and $f, g \in C^\infty(M)$.

ii. $C'_{t_0}[fg] = f(C(t_0))C'_{t_0}[g] + g(C(t_0))C'_{t_0}[f]$, for all $f, g \in C^\infty(M)$.

The properties of tangent vector to a curve lead to the following definition.

Definition 2.9. Let p be a point in a manifold M, a tangent vector to M at p is a map, v_p of $C^\infty(M)$ into R such that

$$v_p[af + bg] = av_p[f] + bv_p[g]$$

$$v_p[fg] = fv_p[g] + gv_p[f]$$

for all $a, b \in R$ and $f, g \in C^\infty(M)$.

Definition 2.10. [17] The tangent space to a manifold M at $p \in M$ is the set of all tangent vectors to M at the point p, and it is denoted by T_pM.

Remark 2.2. The tangent space is a real vector space with the operations defined by

$$(v_p + w_p)[f] = v_p[f] + w_p[f]$$

$$(av_p)[f] = a(v_p[f]).$$

for $v_p, w_p \in T_pM$, $f \in C^\infty(M)$, and $a \in R$.

Definition 2.11. [17] The tangent bundle of a manifold M, denoted by TM, is the set of all tangent vectors at all points of M, that is,

$$TM = \bigcup_p T_pM.$$

Hence, a point of TM is a vector v that is tangent to M at some point $p \in M$. If a manifold M is an n-dimensional, then the manifold TM is a $2n$-dimensional.

If (U, ϕ) is a chart on M, with coordinates $x^1, x^2, \ldots, x^n$ and $p \in U$, then the tangent vectors, $(\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial x^2}), \ldots, (\frac{\partial}{\partial x^n})_p$, are defined by

$$\left(\frac{\partial}{\partial x^i}\right)_p [f] = D_i(f \circ \phi^{-1})|_{\phi(x)}$$

where D_i is the partial derivative with respect to the ith argument; that is,

$$\begin{aligned}\left(\frac{\partial}{\partial x^i}\right)_p [f] &= \lim_{t \to 0} \frac{1}{t}[(f \circ \phi^{-1})(x^1(p), \ldots, x^i(p) + t, \ldots, x^n(p)) \\ &\quad -(f \circ \phi^{-1})(x^1(p), \ldots, x^i(p), \ldots, x^n(p))].\end{aligned}$$

Take $f = x^j$ in the previous formula, and noting that

$$(x^j \circ \phi^{-1})(x^1(p), \ldots, x^i(p), \ldots, x^n(p)) = (x^j \circ \phi^{-1})(\phi(p)) = x^j(p)$$

and,

$$(x^j \circ \phi^{-1})(x^1(p), \ldots, x^i(p) + t, \ldots, x^n(p)) = \begin{cases} x^j, & \text{if} \quad i \neq j \\ x^j + t, & \text{if} \quad i = j \end{cases}$$

(for t is sufficiently small, so that all the points lie in U)

$$\left(\frac{\partial}{\partial x^i}\right)_p [x^j] = \delta_i^j = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases}.$$

Theorem 2.1. *If (U, ϕ) is a chart on M and $p \in U$, the set $\{(\frac{\partial}{\partial x^i})_p\}_{i=1}^n$ is a basis for T_pM.*

If we replace each vector space T_pM with its dual space T_p^*M, we obtain a new $2n$-manifold called the cotangent bundle, denoted by T^*M. The dual basis to $\frac{\partial}{\partial x^i}$ is denoted by dx^i.

Thus, relative to a choice of local coordinates we get the basic formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

for any smooth function $f : M \to R$.

2.3 Differential forms

The main idea of differential forms is to provide a generaliztion of the basic operations of vector calculus, div, grad, and curl, and the integral theorems of Green, Gauss, and stokes to a manifold of certain dimension. They are applied in some areas of physics, mainly in classical mechanics, and of mathematics, such as differential equation, and differential geometry. A simple example of differential 0-form is a real-valued function.

Definition 2.12. [17][**Multilinear map**] A map $\beta : V \times V \times \ldots \times V$ (k-factor) $\to R$ is called a multilinear if it is linear in each of its factors; that is, for all $v_1, v_2, \ldots, v_k \in V$,

$$\beta(v_1, v_2, \ldots, av_i + bv_i', \ldots, v_k) = a\beta(v_1, v_2, \ldots, v_i, \ldots, v_k) + b\beta(v_1, v_2, \ldots, v_i', \ldots, v_k),$$

$\forall 1 \leq i \leq k$.

Definition 2.13. [17][**Skew map**] A k-multilinear map $\beta : V \times V \times \ldots \times V \to R$ is called a skew (or alternating) if it changes sign whenever two of its arguments are interchanged; that is, for all $v_1, v_2, \ldots, v_k \in V$,

$$\beta(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\beta(v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Definition 2.14. [7][**Tensor**] A tensor of type (k, l) at a point x in a manifold M is a multilinear map which takes k vectors and l covectors and gives a real number,

$$T_x : \underbrace{T_xM \times T_xM \times \ldots \times T_xM}_{\text{k-times}} \times \underbrace{T_x^*M \times T_x^*M \times \ldots \times T_x^*M}_{\text{l-times}} \to R.$$

Definition 2.15. [17][**Differential** 1**-form**] A differential 1-form on a manifold M is a linear map ω that is defined on a tangent space of M at a point m

$$\omega(m) : T_mM \to R.$$

Definition 2.16. [17][**Differential** 2**-form**] A differential 2-form on a manifold M is an alternate bilinear map ω that is defined on a tangent space of M at a ponit m

$$\omega(m) : T_mM \times T_mM \to R$$

Definition 2.17. [17][**Differential** k**-form**] A differential k-form on a manifold M is an alternate k-multilinear map ω that is defined on a tangent space of M at m

$$\omega(m) : \underbrace{T_mM \times \ldots \times T_mM}_{\text{k-times}} \to R.$$

Note that a differential k-form is a tensor of type $(k, 0)$ with a skew-symmetry assumption.

A differential k-form on $\mathbb{R}^n$ is a map ω which has the following form

$$\omega(x) = \sum_{i_1<\ldots<i_k} f_{i_1,\ldots,i_k}(x)(dx_{i_1}\wedge\ldots\wedge dx_{i_k})_x, \qquad i_j\in\{1,\ldots,n\},$$

where the $f_{i_1,\ldots,i_k}$ are differentiable real-valued functions on $\mathbb{R}^n$, such that

$$dx_i\wedge dx_j = -dx_j\wedge dx_i.$$

2.4 Tensor and Exterior Products

Definition 2.18. [7][**Tensor Product**] Let T_1 and T_2 be two tensors at a point x on a manifold M of types (k_1,l_1) and (k_2,l_2), respectively. Then, the tensor product $T_1\otimes T_2$ is the tensor at $x\in M$ of type (k_1+k_2,l_1+l_2) defined by

$$\begin{aligned} T_1\otimes T_2(v_1,\ldots,v_{k_1+k_2},w_1,\ldots,w_{l_1+l_2}) &= T_1(v_1,\ldots,v_{k_1},w_1,\ldots,w_{l_1}) \\ &\quad \times T_2(v_{k_1+1},\ldots,v_{k_1+k_2},w_{l_1+1},\ldots,w_{l_1+l_2}) \end{aligned}$$

for all vectors $v_1,\ldots,v_{k_1+k_2}\in T_xM$ and for all covectors $w_1,\ldots,w_{l_1+l_2}\in T_x^*M$.

Definition 2.19. [17][**Alternation Operator A**] If α is $(p,0)$-tensor, define the alternation operator A acting on α by

$$A(\alpha)(v_1,v_2,\ldots,v_p) = \frac{1}{p!}\sum_{\sigma\in S_p} sgn(\sigma)\alpha(v_{\sigma(1)},v_{\sigma(2)},\ldots,v_{\sigma(p)})$$

where the $sgn(\sigma)$ is the sign of the permutation σ :

$$sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd,} \end{cases}$$

and S_p is the group of all permutations of the set $\{1,2,\ldots,p\}$. The operator

A skew-symmetrizes p-multilinear maps.

Definition 2.20. [17][**Exterior Product**] If α is a k-form and β is an l-form on M, their exterior product $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta).$$

Example 2.2. If α and β are 1-forms ($(1,0)$-tensors), then

$$\alpha \wedge \beta(v_1, v_2) = \frac{2!}{1!1!} A(\alpha \otimes \beta)(v_1, v_2)$$

where

$$\begin{aligned} A(\alpha \otimes \beta)(v_1, v_2) &= \frac{1}{2!} \sum_{\sigma \in S_2} sgn(\sigma)(\alpha \otimes \beta)(v_{\sigma(1)}, v_{\sigma(2)}) \\ &= \frac{1}{2!} \sum_{\sigma \in \{(1)(2),(12)\}} sgn(\sigma)(\alpha \otimes \beta)(v_{\sigma(1)}, v_{\sigma(2)}) \\ &= \frac{1}{2} sgn((1)(2))(\alpha \otimes \beta)(v_1, v_2) + \frac{1}{2} sgn((12))(\alpha \otimes \beta)(v_2, v_1) \\ &= \frac{1}{2}\alpha(v_1)\beta(v_2) - \frac{1}{2}\alpha(v_2)\beta(v_1) \end{aligned}$$

Thus,

$$\alpha \wedge \beta(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

Example 2.3. Let α be a 2-form ($(2,0)$-tensor) and β be a 1-form ($(1,0)$-tensor), then

$$\alpha \wedge \beta(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_2, v_3)\beta(v_1) + \alpha(v_3, v_1)\beta(v_2).$$

The exterior product of the differential forms has the following properties.

Proposition 2.2. *[17] Let α be a k-form, β be an s-form, γ_1 and γ_2 are r-forms and a, b are real-valued functions.*
Then:

i. ***The exterior product is associative:*** $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

ii. ***The exterior product is homogeneous:*** $(a\alpha) \wedge \beta = \alpha \wedge (a\beta) = a(\alpha \wedge \beta)$.

iii. ***The exterior product is distributive:***

$$\alpha \wedge (a\gamma_1 + b\gamma_2) = a\alpha \wedge \gamma_1 + b\alpha \wedge \gamma_2$$

$$(a\gamma_1 + b\gamma_2) \wedge \beta = a\gamma_1 \wedge \beta + b\gamma_2 \wedge \beta.$$

iv. ***The exterior product is anticommutative:*** $\alpha \wedge \beta = (-1)^{ks}\beta \wedge \alpha$.

v. *If k is odd then $\alpha \wedge \alpha = 0$. But, it is not true that $\alpha \wedge \alpha = 0$ in general.*

vi. *For any k-form ω, $(\omega)^s = \underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{s\text{-times}}$ is a (ks)-form.*

Proposition 2.3. *[16] Differential $1-$forms $\omega_1, \omega_2, \ldots, \omega_r$ are linearly dependent if and only if their exterior product vanishes; i.e.,*

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r = 0$$

Proof. Let $\omega_1, \omega_2, \ldots, \omega_r$ be linearly dependent $1-$forms. Without loss of generality, assume that ω_1 can be expressed as a linear combination of the others,

$$\omega_1 = a^2\omega_2 + a^3\omega_3 + \ldots + a^r\omega_r$$

Using the properties of the exterior product, we obtain

$$\begin{aligned} \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r &= (\sum_{i=2}^{r} a^i\omega_i) \wedge \omega_2 \wedge \ldots \wedge \omega_r \\ &= a^2\omega_2 \wedge \omega_2 \wedge \ldots \wedge \omega_r + a^3\omega_3 \wedge \omega_2 \wedge \ldots \wedge \omega_r + \ldots + a^r\omega_r \wedge \omega_2 \wedge \ldots \wedge \omega_r \\ &= 0 + 0 + \ldots + 0 \\ &= 0. \end{aligned}$$

Conversely, by contradiction, suppose $\omega_1, \omega_2, \ldots, \omega_r$ are linearly independent $1-$forms, then there exists a basis $\{e_i\}$ such that

$$e_1 = \omega_1, e_2 = \omega_2, \ldots, e_r = \omega_r$$

But,

$$e_1 \wedge e_2 \wedge \ldots \wedge e_r = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r = 0$$

Which is a contradiction, since $e_1 \wedge e_2 \wedge \ldots \wedge e_r$ is a basis vector, it cannot vanish. □

2.5 Examples of Algebraic Computation of The Exterior Product

Example 2.4. Let $\omega = x^1dx^1 + x^3dx^2 + x^2dx^3$ be a 1-form in $\mathbb{R}^3$ and $\phi = x^1dx^1 \wedge dx^2 + x^2dx^1 \wedge dx^3$ be a 2-form in $\mathbb{R}^3$. Using the fact, $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i, \quad \forall i, j \in \{1, 2, 3\}$.

$$\begin{aligned}
\omega \wedge \phi &= (x^1dx^1 + x^3dx^2 + x^2dx^3) \wedge (x^1dx^1 \wedge dx^2 + x^2dx^1 \wedge dx^3) \\
&= \cancelto{0}{(x^1)^2dx^1 \wedge dx^1} \wedge dx^2 + \cancelto{0}{x^1x^2dx^1 \wedge dx^1} \wedge dx^3 \\
&\quad + \cancelto{0}{x^1x^3dx^2 \wedge dx^1 \wedge dx^2} + x^2x^3dx^2 \wedge dx^1 \wedge dx^3 \\
&\quad + x^1x^2dx^3 \wedge dx^1 \wedge dx^2 + \cancelto{0}{(x^2)^2dx^3 \wedge dx^1 \wedge dx^3} \\
&= (x^1x^2 - x^2x^3)dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Example 2.5. Let $\alpha = x^1dx^1 + x^2dx^2$ be a 1-form in $\mathbb{R}^3$ and $\beta = x^1x^3dx^1 \wedge$

$dx^3 + x^2x^3dx^2 \wedge dx^3$ be a 2-form in $\mathbb{R}^3$.

$$\begin{aligned}
\alpha \wedge \beta &= (x^1dx^1 + x^2dx^2) \wedge (x^1x^3dx^1 \wedge dx^3 + x^2x^3dx^2 \wedge dx^3) \\
&= (x^1)^2x^3\cancelto{0}{dx^1 \wedge dx^1} \wedge dx^3 + x^1x^2x^3dx^1 \wedge dx^2 \wedge dx^3 \\
&\quad +x^1x^2x^3dx^2 \wedge dx^1 \wedge dx^3 + (x^2)^2x^3\cancelto{0}{dx^2 \wedge dx^2} \wedge x^3 \\
&= (x^1x^2x^3 - x^1x^2x^3)dx^1 \wedge dx^2 \wedge x^3 = 0.
\end{aligned}$$

Note that, $\beta = \alpha \wedge \gamma$, where $\gamma = x^3dx^3$ is a 1-form in $\mathbb{R}^3$. So, $\alpha \wedge \beta = \alpha \wedge \alpha \wedge \gamma = 0$.

Example 2.6. Let $\omega = x^1dx^1 \wedge dx^2 + x^2dx^3 \wedge dx^4$ be a 2-form in $\mathbb{R}^4$.Then

$$\omega \wedge \omega = 2x^1x^2dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$$

2.6 Exterior Derivative

We now define the exterior derivative of differential forms.

Definition 2.21. [17] The exterior derivative of a differential k-form α on a manifold M is the differential $(k+1)$-form on M, denoted by $d\alpha$.

The exterior derivative can be determined by the following proposition.

Proposition 2.4. *[17] There is a unique linear operator d from the set of k-forms on a manifold M; $\Lambda^k(M)$, to the set of $(k+1)$-forms on M; $\Lambda^{k+1}(M)$, such that*

$$d : \Lambda^k(M) \rightarrow \Lambda^{k+1}(M)$$

i. *If α is a* 0*-form; that is, $\alpha \in C^\infty(M)$, then $d\alpha$ is the* 1*-form.*

$$d\alpha = \sum_{i=1}^{n} \frac{\partial \alpha}{\partial x^i} dx^i.$$

ii. *d is a linear operation, that is, for all real numbers a and b,*

$$d(a\alpha_1 + b\alpha_2) = ad\alpha_1 + bd\alpha_2$$

.

iii. *$d^2\alpha = 0$, that is, $d(d\alpha) = 0$ for any k-form α.*

iv. *If $\alpha = \sum_{i_1<\ldots<i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ is a k-form in $\mathbb{R}^n$, then the coordinate expression for the exterior derivative is*

$d\alpha = \sum \frac{\partial f_{i_1,\ldots,i_k}}{\partial x^j} dx^j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ (sum on all j and $i_1 < \ldots < i_k$)

Definition 2.22. [17] A differential k-form ω is called closed if $d\omega = 0$, and exact if there exists a differential $(k-1)$-form α such that $d\alpha = \omega$.

Definition 2.23. [8] Let M be a differentiable manifold. A one-parameter group of transformations; φ, on M, is a differentiable map from $M \times \mathbb{R}$ onto M such that $\varphi(x, 0) = x$ and $\varphi(\varphi(x, t), s) = \varphi(x, t+s)$ for all $x \in M, t, s \in \mathbb{R}$. The infinitesimal generator of φ is the vector field X such that $X_x = (\varphi(x, 0)^{'})$.

Lemma 2.5. *[17][**Poincaré's Lemma**] A closed form is locally exact; that is, if the differential k-form ω is closed($d\omega = 0$) then there exists a differential $(k-1)$-form α such that $\omega = d\alpha$ on some neighborhood of each point.*

The proof can be found in [8].

Theorem 2.6. *[13][**Cartan's Magic Formula**] The exterior derivative of the exterior product of a differential p-form ω and a differential q-form φ is given by*

$$d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^p \omega \wedge d\varphi$$

2.7 Examples of Algebraic Computations of The Exterior Derivative

Example 2.7. Let $\omega = \sum_{i=1}^{n} \omega_i dx^i$ be 1-form in $\mathbb{R}^n$, then

$$\begin{aligned} d\omega &= d(\sum_{i=1}^{n} \omega_i dx^i) \\ &= \sum_{i=1}^{n} d\omega_i \wedge dx^i \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i \\ &= \sum_{1\leq i<j\leq n}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i + \sum_{1\leq j<i\leq n}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i \\ &= -\sum_{1\leq i<j\leq n}^{n} \frac{\partial \omega_i}{\partial x^j} dx^i \wedge dx^j + \sum_{1\leq i<j\leq n}^{n} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{1\leq i<j\leq n}^{n} (\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}) dx^i \wedge dx^j. \end{aligned}$$

Example 2.8. Let $\omega = \frac{x^2}{(x^1)^2}dx^1 + \frac{1}{x^1}dx^2$ be a differential 1-form in $\mathbb{R}^n$, then

$$\begin{aligned} d\omega &= \left(\frac{2x^2}{(x^1)^3}dx^1 + \frac{1}{(x^1)^2}dx^2\right) \wedge dx^1 + \frac{-1}{(x^1)^2}dx^1 \wedge dx^2 \\ &= \frac{2x^2}{(x^1)^3}dx^1 \wedge dx^1 + \frac{1}{(x^1)^2}dx^2 \wedge dx^1 + \frac{-1}{(x^1)^2}dx^1 \wedge dx^2 = 0 \end{aligned}$$

So, ω is a closed 1-form. By **Poincaré's Lemma**, there exists a 0-form $f = \frac{x^2}{x^1}$ such that $df = \omega$ on some neighborhood of each point.

Example 2.9. Let $\omega = \sum_{1 \le i < j \le n} \omega_{i,j} dx^i \wedge dx^j$ be a differential 2-form in $\mathbb{R}^n$, then

$$\begin{aligned} d\omega &= \sum_{1 \le i < j \le n} d\omega_{i,j} \wedge dx^i \wedge dx^j \\ &= \sum_{1 \le i < j \le n} \sum_{k=1}^{n} \frac{\partial \omega_{i,j}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \\ &= \sum_{1 \le k < i < j \le n} \frac{\partial \omega_{i,j}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j + \sum_{1 \le i < k < j \le n} \frac{\partial \omega_{i,j}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \\ &\quad + \sum_{1 \le i < j < k \le n} \frac{\partial \omega_{i,j}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \\ &= \sum_{1 \le i < j < k \le n} \frac{\partial \omega_{j,k}}{\partial x^i} dx^i \wedge dx^j \wedge dx^k + \sum_{1 \le i < j < k \le n} \frac{\partial \omega_{i,k}}{\partial x^j} dx^j \wedge dx^i \wedge dx^k \\ &\quad + \sum_{1 \le i < j < k \le n} \frac{\partial \omega_{i,j}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \\ &= \sum_{1 \le i < j < k \le n} \left(\frac{\partial \omega_{j,k}}{\partial x^i} - \frac{\partial \omega_{i,k}}{\partial x^j} + \frac{\partial \omega_{i,j}}{\partial x^k}\right) dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

Example 2.10. Let $\alpha = xyz^2 dx \wedge dy + ydx \wedge dz$ be 2-form in $\mathbb{R}^3$, then

$$\begin{aligned}
d\alpha &= (yz^2dx + xz^2dy + 2xyzdz) \wedge dx \wedge dy + dy \wedge dx \wedge dz \\
&= \cancelto{0}{yz^2dx \wedge dx} \wedge dy + \cancelto{0}{xz^2dy \wedge dx} \wedge dy + 2xyzdz \wedge dx \wedge dy + dy \wedge dx \wedge dz \\
&= 2xyzdz \wedge dx \wedge dy + dy \wedge dx \wedge dz \\
&= (2xyz - 1)dx \wedge dy \wedge dz.
\end{aligned}$$

2.8 Interior Product and Lie Derivative

Definition 2.24. [8][**Interior Product**] Let ω be a differential k-form and X be a vector field on a manifold M. Define the interior product $\iota_X\omega$ (sometimes called a contraction of X and ω , and written $X \lrcorner \omega$)

$$\iota_X = \Lambda^k(M) \to \Lambda^{k-1}(M)$$

by

$$\iota_X\omega(Y_1, Y_2, \ldots, Y_{k-1}) = \omega(X, Y_1, Y_2, \ldots, Y_{k-1}).$$

The interior product of differential forms has the following properties.

Proposition 2.7. *[8] Let ω be a differential k-form defined on a manifold M and α_1, α_2 be two differential s-forms on M, and X, Y be two vector fields on M, then :*

i. $\iota_X\omega$ *is a differential* $(k-1)$*-form on* M*.*

ii. ι_X *is linear map, that is, for any real numbers* c_1, c_2,

$$\iota_X(c_1\alpha_1 + c_2\alpha_2) = c_1\iota_X\alpha_1 + c_2\iota_X\alpha_2.$$

iii.

$$\iota_Y\iota_X\omega = -\iota_X\iota_Y\omega.$$

iv. The interior product of the exterior product,

$$\iota_X(\omega \wedge \alpha_1) = \iota_X\omega \wedge \alpha_1 + (-1)^k\omega \wedge \iota_X\alpha_1$$

Definition 2.25. Let M be a differentiable manifold. A one-parameter group of transformations, $\varphi_t(x)$, on M, is a differentiable map from $M \times \mathbb{R}$ onto M such that $\varphi_0(x) = x$ and $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$ for all $x \in M, t, s \in \mathbb{R}$. The infinitesimal generator of φ is the vector field X such that $X = \varphi_t(x)'|_{t=0}$.

Definition 2.26. [5] Let X be a vector field on a manifold M and ω be a differential k-form defined on M. The Lie derivative of ω with respect to X is the object whose value at $x \in M$ is:

$$\mathcal{L}_X\omega = \lim_{t\to 0} \frac{\phi_t^*(\omega|_{\phi_t(x)}) - \omega|_x}{t} = \frac{d}{dt}\bigg|_{t=0} \phi_t^*(\omega|_{\phi_t(x)})$$

where $\phi_t(x)$ is the flow of the vector field X and $\phi_t^*(x)$ refers to the pull-back of $\phi_t(x)$, defined by

$$\phi_t^*(\omega) = \omega(\phi_t(x)).$$

Proposition 2.8. *[5] Let X be a vector field on a manifold M and ω, φ be two differential k-forms defined on M, then the Lie derivative has the following properties:*

i. *$\mathcal{L}_X\omega$ is of the same degree as ω.*

ii. *The linearity of Lie derivative, that is, for any real numbers c_1 and c_2,*

$$\mathcal{L}_X(c_1\omega + c_2\varphi) = c_1\mathcal{L}_X\omega + c_2\mathcal{L}_X\varphi.$$

iii. *Commutation with the differential,*

$$d\mathcal{L}_X\omega = \mathcal{L}_X d\omega.$$

iv. *The Lie derivative of the exterior product,*

$$\mathcal{L}_X(\omega \wedge \varphi) = \mathcal{L}_X\omega \wedge \varphi + \omega \wedge \mathcal{L}_X\varphi.$$

v. $\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$.

where ι is the interior product between ω and X and "d" is the exterior derivative.

Definition 2.27. [5] Given a differential 1-form $\omega = \sum_{i=1}^{n} \omega_i dx^i$ defined on a cone $C \subset \mathbb{R}^n$, we say that the ω is k-homogeneous if the functions $\omega_i, i = 1, \ldots, n$ are k-homogeneous for all $x \in C$.

Theorem 2.9. *[5] The differential form $\omega = \sum_{i=1}^{n} \omega_i dx^i$ is k-homogeneous if and only if*

$$\mathcal{L}_X \omega = (k+1)\omega$$

where $X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} \in TR^n$.

Proof. Let ω and X be defined as above, and using the property v.; that is,

$$\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega).$$

We calculate each term on the right hand side.

Since,

$$d\omega = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i.$$

Then,

$$\iota_X d\omega = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} x^j dx^i - \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} x^i dx^j. \tag{2.2}$$

Similarly, we have

$$\iota_X \omega = \sum_{i=1}^{n} \omega_i(x) x^i.$$

Then,

$$d(\iota_X \omega) = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} x^i dx^j + \sum_{i=1}^{n} \omega_i dx^i. \tag{2.3}$$

From (3.3) and (3.4), we get

$$\begin{aligned}\mathcal{L}_X\omega &= \sum_{i,j=1}^{n}\frac{\partial\omega_i}{\partial x^j}x^j dx^i - \sum_{i,j=1}^{n}\frac{\partial\omega_i}{\partial x^j}x^i dx^j + \sum_{i,j=1}^{n}\frac{\partial\omega_i}{\partial x^j}x^i dx^j + \sum_{i=1}^{n}\omega_i dx^i \\ &= \sum_{i,j=1}^{n}\frac{\partial\omega_i}{\partial x^j}x^j dx^i + \sum_{i=1}^{n}\omega_i dx^i.\end{aligned}$$

By Euler's Theorem, we conculde that $\omega(x)$ is k-homogeneous if and only if

$$\mathcal{L}_X\omega = (k+1)\omega.$$

This completes the proof. □

Corollary 2.10. *[5] If ω is k-homogeneous differential 1-form, then*

i. $\mathcal{L}_X\omega \wedge \omega = 0.$

ii. $\mathcal{L}_X(\omega \wedge d\omega) = 2(k+1)\omega \wedge d\omega.$

iii. $\mathcal{L}_X d\omega = (k+1)d\omega.$

Theorem 2.11. *[5] Let ω be a C^1 differential m-form. Then ω is k-homogeneous if and only if*

$$\mathcal{L}_X\omega = (k+m)\omega.$$

let ω be a differential 1-form. Define a sequence of differential forms:

$$\omega_1 = \omega, \qquad \omega_2 = d\omega, \qquad \omega_3 = \omega\wedge d\omega, \qquad \omega_4 = d\omega\wedge d\omega$$
$$\omega_5 = \omega\wedge d\omega\wedge d\omega, \ldots, \text{etc.}$$

Definition 2.28. [5] The rank of a differential 1-form ω at a point x in a manifold M is the integer $0 \leq r(x) \leq n$ such that $\omega_i(x) \neq 0$ for $i \leq r$, whereas $\omega_i(x) = 0$ for all $i > r$. Moreover, ω is called regular if $r(x)$ is fixed for all x.

Theorem 2.12. *[5][**Darboux**] Suppose ω is a differential 1-form of constant rank r on a manifold M. Then, there exist local coordinates $x = (x^1, x^2, \ldots, x^n)$ such that ω has the canonical form:*

$$\omega = \begin{cases} x^1 dx^2 + \ldots + x^{2s-1} dx^{2s}, & r = 2s \\ x^1 dx^2 + \ldots + x^{2s-1} dx^{2s} + dx^{2s+1}, & r = 2s+1 \end{cases}$$

Definition 2.29. [16] A subring $\mathcal{I}$ which is a subset of the set of k-forms on a manifold M; $\Lambda^k(M)$, is called ideal if:

a) $\alpha \in \mathcal{I}$ implies $\alpha \wedge \beta \in \mathcal{I}$ for all $\beta \in \Lambda^k(M)$.

b) $\alpha \in \mathcal{I}$ implies that all its components in $\Lambda^k(M)$ are contained in $\mathcal{I}$.

Definition 2.30. [16][**Differential Ideal**] An ideal $\mathcal{I} \subset \Lambda^k(M)$ satisfying $d\mathcal{I} \subset \mathcal{I}$ is called a differential ideal, where

$$d\mathcal{I} = \{d\alpha | \alpha \in \mathcal{I}\}.$$

Definition 2.31. [16][**Forbenius Condition**] Let $\mathcal{I}$ be a differential ideal having as generators the linear forms $\alpha^1, \ldots, \alpha^{n-r}$ of degree one, the condition that $\mathcal{I}$ is closed means

(F) $\qquad d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{n-r}, 1 \leq i \leq n-r.$

The condition (F) is called the Forbenius condition.

Theorem 2.13. *[15][**Forbenius Theorem**] Let $\mathcal{I}$ be a differential ideal having as generators the linear forms $\alpha^1, \ldots, \alpha^{n-r}$ of degree one, so that the Forbenius condition is satisfied. In a sufficiently small neighborhood there is a coordinate system $y^1, \ldots, y^n$ such that $\mathcal{I}$ is generated by $dy^{r+1}, \ldots, dy^n$.*

The proof can be found in [15].

Chapter 3

Convex Darboux Theorem

There are many applications in which we need to write differential forms as a linear combination of gradients. In [15], Darboux found the necessary and sufficient condition that guarantees this combination in a neighborhood; $\mathcal{U}$, of $\bar{x}$ in $\mathbb{R}^n$. However, some economic applications require an additional restriction on the coefficients to be positive functions and the coordinates to be convex functions. There were several attempts to find a necessary and sufficient condition that guarantees the positivity of the coefficients and the convexity of the coordinates.

In [9], Chiappori and Ekeland gave the result when a 1-form ω is analytic. Later on, in [21], Zakalyukin gave the result when ω is smooth. In [14], it has been shown that their results are false by Ekeland and Nirenberg by giving a counterexample and they found a necessary and sufficient condition that guarantees the positivity of the coefficients and the convexity of the coordinates.

3.1 Introduction

Let $\omega = \sum_{i=1}^{n} \omega_i dx^i$ be a smooth differential 1-form defined on a neighborhood; $\mathcal{U}$, of the origin in $\mathbb{R}^n$. The problem of finding necessary and sufficient condition to decompose the smooth differential 1-form ω defined on $\mathcal{U}$ into the sum

$$\omega = f^1 dg_1 + \ldots + f^k dg_k \tag{3.1}$$

has been solved by Ekeland and Nirenberg using Exterior Differential Calculus, where the f^l are positive functions and the g_l are strictly convex functions.

By a classic result in exterior differential calculus; Darboux Theorem, if ω has rank $2k$; that is,

$$\omega \wedge (d\omega)^{k-1} \neq 0 \quad \text{and} \quad \omega \wedge (d\omega)^k = 0 \quad \text{on} \quad \mathcal{U}$$

then (3.1) holds. If ω satisfies (3.1), then

$$d\omega = \sum_{l=1}^{k} df^l \wedge dg_l$$

and

$$(d\omega)^k = k! df^1 \wedge dg_1 \wedge df^2 \wedge dg_2 \ldots \wedge df^k \wedge dg_k$$

Hence,

$$\omega \wedge (d\omega)^k = 0$$

But, Darboux Theorem does not give any guarantee for positiveness of the coefficients and convexity of the coordinates. In [9], Chiappori and Ekeland found a necessary and sufficient condition to decompose the analytical differential 1-form ω defined on a neighborhood; $\mathcal{U}$, of origin into the sum

$$\omega = \sum_{l=1}^{k} f^l dg_l$$

where the coefficients are positive functions and the coordinates are convex functions.

Chiappori and Ekeland condition: There is some neighborhood of the origin where the matrix $(\omega_{i,j})$ is the sum of two matrices, a positive definite one and another one of rank k, where

$$\omega_{i,j} = \frac{\partial \omega_i}{\partial x^j}.$$

In [21], Zakalyukin was interested in finding a necessary and sufficient condition to decompose a smooth (non-analytical) differential 1-form ω defined on a neighborhood; $\mathcal{U}$, of the origin into the sum

$$\omega = \sum_{l=1}^{k} f^l dg_l$$

where the coefficients f^l are positive functions and the coordinates g_l are convex functions. He introducedthe space $A_2(\omega)$ of all tangent vector fields ξ such that

$$\iota_\xi \omega = 0.$$

$$\iota_{(\xi,\eta)} d\omega = 0, \quad \forall \eta.$$

Zakalyukin Condition: In addition to Chiappori and Ekeland condition, he requires the following condition: There is some neighborhood of the origin where the matrix $(\omega_{i,j}) + (\omega_{j,i})$ is positive definite on $A_2(\omega)$.

In [14], Ekeland and Nirenberg found a counterexample of the previous results.

Example 3.1. [Counterexample of Chiappori and Ekeland condition]
Consider $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4 and the differential 1-form

$$\omega = (1 + x^1 + x^4)dx^1 + x^2 dx^2 + (x^2 + x^3)dx^3.$$

Then,

$$\begin{aligned} \omega \wedge d\omega &= \omega \wedge \left(dx^4 \wedge dx^1 + dx^2 \wedge dx^3\right) \\ &= (1 + x^1 + x^4)dx^1 \wedge dx^2 \wedge dx^3 + x^2 dx^2 \wedge dx^4 \wedge dx^1 + (x^2 + x^3)dx^3 \wedge dx^4 \wedge dx^1 \\ &\neq 0 \end{aligned}$$

and

$$\omega \wedge (d\omega)^2 = 0.$$

Hence, $k = 2$. Moreover,

$$\omega_{i,j} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = I + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus, we can write $\omega_{i,j}$ as the sum of a positive definite matrix and a matrix of rank 2, which means the Chiappori and Ekeland condition holds. But, the problem has no solution. Assume otherwise, there exist smooth functions f^1, f^2, g_1, g_2 such that

$$\omega = f^1 dg_1 + f^2 dg_2.$$

where f^1, f^2 are positive functions and g_1, g_2 are strictly convex functions. Then g_1 satisfies:

$$dg_1 \wedge \omega \wedge d\omega = dg_1 \wedge f^2 dg_2 \wedge d\omega = 0. \tag{3.2}$$

On the other hand:

$$\omega\wedge d\omega = (1+x^1+x^4)dx^1\wedge dx^2\wedge dx^3+x^2dx^2\wedge dx^4\wedge dx^1+(x^2+x^3)dx^3\wedge dx^4\wedge dx^1$$

Substituting into equation (3.2), we get

$$-(1+x^1+x^4)\frac{\partial g_1}{\partial x^4}+x^2\frac{\partial g_1}{\partial x^3}-(x^2+x^3)\frac{\partial g_1}{\partial x^2}=0.$$

In particular, on the plane $x^2 = x^3 = 0$, we have

$$(1+x^1+x^4)\frac{\partial g_1}{\partial x^4}=0.$$

So, $\frac{\partial g_1}{\partial x^4} = 0$ on the plane $x^2 = x^3 = 0$. Hence, g_1 cannot be strictly convex.

Example 3.2. [Counterexample of Zakalyukin condition]

Consider $\mathbb{R}^5$ with coordinates x^1, x^2, x^3, x^4, x^5 and the differential 1-form

$$\omega = -x^2dx^1 + x^1dx^2 + (1+x^3)dx^3 + (1+x^4)dx^4 + (1+x^5)dx^5.$$

Then,

$$\omega\wedge d\omega = 2\left(dx^1\wedge dx^2\right)\wedge\left((1+x^3)dx^3+(1+x^4)dx^4+(1+x^5)dx^5\right)\neq 0$$

and

$$\omega\wedge(d\omega)^2 = 0.$$

Hence, $k = 2$. Darboux condition holds. Moreover,

$$\omega_{i,j} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I + \begin{pmatrix} -1 & -1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So, Chiappori and Ekeland condition holds as well. At the origin, $\xi \in A_2(\omega)$ means that

$$\langle \omega | \xi \rangle = 0 \implies \xi^3 + \xi^4 + \xi^5 = 0.$$

$$\langle d\omega | (\xi, \eta) \rangle = 0 \implies 2(\xi^1\eta^2 - \xi^2\eta^1) = 0, \quad \forall (\eta^1, \eta^2).$$

So, $(\xi^1\eta^2 - \xi^2\eta^1) = 0, \quad \forall(\eta^1, \eta^2)$ means that $\xi^1 = \xi^2 = 0$. Thus, the matrix

$$\omega_{i,j} + \omega_{j,i} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

is positive definite on the space $A_2(\omega) = \{(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5) | \xi^1 = \xi^2 = 0\}$. So the Zakalyukin condition is satisfied also.
We claim that ω cannot be written in the form:

$$\omega = adu + bdv$$

where a and b are positive functions and u and v are convex functions. Assume otherwise. In particular, on the plane $x^3 = x^4 = x^5 = 0$, we have:

$$a\frac{\partial u}{\partial x^1} + b\frac{\partial v}{\partial x^1} = -x^2 \tag{3.3}$$

$$a\frac{\partial u}{\partial x^2} + b\frac{\partial v}{\partial x^2} = x^1 \tag{3.4}$$

We assume that $u(0) = v(0) = 0$, then the Taylor expansion to u, v near the origin in the plane (x^1, x^2)

$$u = c_1x^1 + c_2x^2 + Q_1(x^1, x^2) + o(||x||^2).$$

$$v = d_1 x^1 + d_2 x^2 + Q_2(x^1, x^2) + o(||x||^2).$$

Respectively, where $Q_1(x^1, x^2), Q_2(x^1, x^2)$ are positive definite quadratic forms. Then,

$$\frac{\partial u}{\partial x^1} = c_1 + \frac{\partial Q_1}{\partial x^1} + o(||x||^2), \quad \frac{\partial u}{\partial x^2} = c_2 + \frac{\partial Q_1}{\partial x^2} + o(||x||^2). \tag{3.5}$$

$$\frac{\partial v}{\partial x^1} = d_1 + \frac{\partial Q_2}{\partial x^1} + o(||x||^2), \quad \frac{\partial v}{\partial x^2} = d_2 + \frac{\partial Q_2}{\partial x^2} + o(||x||^2). \tag{3.6}$$

From equations (3.3) and (3.4), we get:

$$a\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1} - \frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}\right) = x^1\frac{\partial v}{\partial x^1} + x^2\frac{\partial v}{\partial x^2} = d_1x^1 + d_2x^2 + 2Q_2(x^1, x^2) + o(||x||^2) \tag{3.7}$$

$$b\left(\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2} - \frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}\right) = x^1\frac{\partial u}{\partial x^1} + x^2\frac{\partial u}{\partial x^2} = c_1x^1 + c_2x^2 + 2Q_1(x^1, x^2) + o(||x||^2) \tag{3.8}$$

At the origin, the right hand sides vanish, and since a and b are positive functions, we must have

$$\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1} - \frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2} = c_2 d_1 - c_1 d_2 = 0$$

This implies that the vectors (c_1, c_2) and (d_1, d_2) are parallel. One or both may vanish, but in any case we can choose $(x^1, x^2) \neq 0$ near the origin so that

$$c_1 x^1 + c_2 x^2 = 0 = d_1 x^1 + d_2 x^2.$$

For such a choice of (x^1, x^2), the right hand sides of equations (3.7) and (3.8) are positive. But, the left hand sides have opposite signs.

3.2 Ekeland-Nirenberg Theorem

In this section, we discuss the Ekeland-Nirenberg Theorem which gives an answer to the following problem.

Problem: Under what conditions can we represent a smooth differential 1-form $\omega = \sum_{i=1}^{n} \omega_i dx^i$ defined on a neighborhood; $\mathcal{U}$, of the origin in $\mathbb{R}^n$, in the form:

$$\omega = \sum_{l=1}^{k} f^l dg_l \tag{3.9}$$

where the f^l are positive functions and the g_l are strictly convex functions? In [14], the previous problem was solved by Ekeland and Nirenberg and gave the following necessay and sufficient condition.

Ekeland-Nirenberg Condition: Consider the subspace of the space of all 1-forms α defined as follow:

$$\mathcal{I} = \{\alpha | \alpha \wedge \omega \wedge (d\omega)^{k-1} \equiv 0\}.$$

There is a k-dimensional subspace V of $\mathcal{I}(0)$, containing $\omega(0)$, such that on $N = V^{\perp}$, the matrix $(\omega_{i,j})(0)$ is symmetric and positive definite.

The Ekeland-Nirenberg condition requires that :

$$\xi^T(\omega_{i,j})(0)\eta = \eta^T(\omega_{i,j})(0)\xi, \quad \forall \xi, \eta \in N.$$

$$\xi^T(\omega_{i,j})(0)\xi > 0, \quad \forall 0 \neq \xi \in N.$$

Where N is the subspace of vectors ξ such that

$$\iota_\xi \alpha = 0, \quad \forall \alpha \in V.$$

Using this condition, Ekeland-Nirenberg stated the following theorem.

Theorem 3.1 (Ekeland-Nirenberg Theorem)**.** *Assume ω is a smooth dif-*

ferential 1-form satisfying $\omega \wedge (d\omega)^{k-1} \neq 0$ *on a neighborhood;* $\mathcal{U}$*, of the origin. Then,* ω *can be decomposed into the sum* $\omega = \sum_{l=1}^{k} f^l dg_l$*, where the* f^l *are positive functions and the* g_l *are convex functions in some neighborhood;* $\mathcal{V} \subset \mathcal{U}$*, of the origin if and only if* $\omega \wedge (d\omega)^k = 0$ *on* $\mathcal{U}$ *and the Ekeland-Nirenberg condition is satisfied at the origin.*

In the following we provide an example.

Example 3.3. Consider $\mathbb{R}^3$ with coordinate system x, y, z and the differential 1-form

$$\omega = (1+y)dx + (1+x)dy + dz.$$

Then we have

$$\omega \wedge d\omega = 0$$

and

$$(\omega_{i,j})(0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

We define the subspace $\mathcal{I}$ of the space of all 1-forms α as follows:

$$\mathcal{I} = \{\alpha | \alpha \wedge \omega = 0\}.$$

So, $\mathcal{I}$ has dimension one and ω is in $\mathcal{I}$, hence

$$\mathcal{I}(0) = \{\omega(0)\}.$$

The Ekeland-Nirenberg condition says that there exists a one dimensional subspace $V \subset \mathcal{I}(0)$ such that the matrix $(\omega_{i,j})(0)$ is positive definite on $V^{\perp}$. Here $\mathcal{I}(0)$ is one dimensional, that means $V = \mathcal{I}(0) = \{\omega(0)\} = \{dx(0) + dy(0) + dz(0)\}$.

$$\{dx(0) + dy(0) + dz(0)\}^{\perp} = \{(\xi^1, \xi^2, \xi^3) | \xi^1 + \xi^2 + \xi^3 = 0\}.$$

Positive definiteness means

$$\begin{pmatrix} \xi^1 & \xi^2 & \xi^3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = 2\xi^1\xi^2 > 0$$

But, the matrix $\omega_{i,j}(0)$ is not positive definite on $V^{\perp}$, since the vector $(1, -1, 0)$ is in $V^{\perp}$. Thus, ω cannot be written as $\omega = du$ where the function u is strictly convex.

3.3 Proof of Convex Darboux Theorem

As mentioned before, the goal of the convex Darboux theorem is to solve the following problem.

Problem: under what conditions can we represent a smooth differential 1-form $\omega = \sum_{i=1}^{n} \omega_i dx^i$ defined on a neighborhood; $\mathcal{U}$, of the origin in $\mathbb{R}^n$, in the form:

$$\omega = \sum_{l=1}^{k} f^l dg_l \tag{3.10}$$

where the f^l are positive functions and the g_l are strictly convex functions?

Theorem 3.2 (**Convex Darboux Theorem).** *Assume ω is a smooth differential 1-form satisfying $\omega \wedge (d\omega)^{k-1} \neq 0$ on a neighborhood; $\mathcal{U}$, of the origin. Then, ω can be decomposed into the sum $\omega = \sum_{l=1}^{k} f^l dg_l$, where the f^l are positive functions and the g_l are strictly convex functions in some neighborhood; $\mathcal{V} \subset \mathcal{U}$, of the origin if and only if $\omega \wedge (d\omega)^k = 0$ on $\mathcal{U}$ and the Ekeland-Nirenberg condition is satisfied at the origin.*

3.3.1 Proof of Necessity

Let $\omega(x)$ be a smooth differential 1-form defined on $\mathbb{R}^n$, satisfying $\omega \wedge (d\omega)^{k-1} \neq 0$ on $\mathcal{U}$. Assume our problem has a solution in some neighborhood; $\mathcal{V} \subset \mathcal{U}$ of the origin; that is, we can repersent $\omega(x)$ in the form:

$$\omega = \sum_{l=1}^{k} f^l dg_l \tag{3.11}$$

in $\mathcal{V} \subset \mathcal{U}$, where the f^l are positive functions and the g_l are strictly convex functions. Then,

$$d\omega = \sum_{l=1}^{k} df^l \wedge dg_l$$

and

$$(d\omega)^k = k! df_1 \wedge dg_1 \wedge \ldots \wedge df_k \wedge dg_k$$

Hence,

$$\omega \wedge (d\omega)^k = 0.$$

It remains to show that the Ekeland-Nirenberg condition holds at the origin. The differential 1-forms $dg_1, dg_2, \ldots, dg_k$ are linearly independent in a neighborhood of the origin. If not, then $\omega(0)$ can be expressed as a linear combination of $k-1$ of them, which is a contradiction with $\omega \wedge (d\omega)^{k-1} \neq 0$ at the origin.

As we defined the subset $\mathcal{I}$ before, $dg_l \in \mathcal{I}, \quad \forall l = 1, 2, \ldots, k$. Since

$$\omega \wedge (d\omega)^{k-1} = \Theta \wedge dg_1 \wedge dg_2 \wedge \ldots \wedge dg_k$$

for some $(k-1)$-form Θ. Thus,

$$dg_l \wedge \omega \wedge (d\omega)^{k-1} = 0, \qquad \forall l = 1, 2, \ldots, k.$$

Let V be the k-dimensional subspace of $\mathcal{I}(0)$ spanned by $dg_1, dg_2 \ldots, dg_k$.

Thus, $\omega(0) = \sum_{l=1}^{k} f^l(0) dg_l(0)$ lies in V. Differentiating (3.11), we find

$$\omega_{i,j} = \sum_{l=1}^{k} \frac{\partial f^l}{\partial x^j}\frac{\partial g_l}{\partial x^i} + \sum_{l=1}^{k} f^l \frac{\partial^2 g_l}{\partial x^i \partial x^j}.$$

Thus, for every $\xi, \eta \in N = V^{\perp}$,

$$\begin{aligned}\sum_{i,j=1}^{n} \omega_{i,j}\xi^i\eta^j &= \sum_{l=1}^{k}\sum_{i,j=1}^{n} \frac{\partial f^l}{\partial x^j}\frac{\partial g_l}{\partial x^i}\xi^i\eta^j + \sum_{l=1}^{k}\sum_{i,j=1}^{n} f^l \frac{\partial^2 g_l}{\partial x^i \partial x^j}\xi^i\eta^j \\ &= \sum_{l=1}^{k}\sum_{i,j=1}^{n} f^l \frac{\partial^2 g_l}{\partial x^i \partial x^j}\xi^i\eta^j.\end{aligned}$$

Since

$$\frac{\partial^2 g_l}{\partial x^i \partial x^j} = \frac{\partial^2 g_l}{\partial x^j \partial x^i}, \qquad \forall l = 1, 1, \ldots, k.$$

the right-hand side is symmetric in ξ and η, therefore the left-hand side is also symmetric. Thus, $\omega_{i,j}(0)$ is symmetric on N. Furthermore, taking $\xi = \eta$, by the assumption that the g_l are strictly convex functions on $\mathbb{R}^n$, we get

$$\sum_{i,j=1}^{n} \frac{\partial^2 g_l}{dx^i dx^j}\xi^i\xi^j > 0, \quad \forall l = 1, 2, \ldots, k, \quad \text{and} \quad 0 \neq \xi \in N.$$

and the f^l are positive functions by the assumption. So, $\sum_{i,j=1}^{n} \omega_{i,j}\xi^i\xi^j$ is a positive number for all $0 \neq \xi \in N$. Hence, $\omega_{i,j}(0)$ is positive definite on N.

3.3.2 Proof of Sufficiency

Firstly, we are going to introduce some algebraic results that will be needed.

Lemma 3.3. *[15] Let $\alpha_1, \ldots, \alpha_{p+1}$ be linearly independent 1-forms and Ω a*

2-form such that

$$\alpha_1 \wedge \ldots \wedge \alpha_{p+1} \wedge \Omega^q = 0$$

for some integers p and q. Then,

$$\alpha_1 \wedge \ldots \wedge \alpha_p \wedge \Omega^{q+1} = 0.$$

Lemma 3.4. *[15] Let $\alpha_1, \ldots, \alpha_{l-1}$ be 1-forms such that $\alpha_1, \ldots, \alpha_{l-1}, \omega$ are linearly independent and satisfy*

$$\alpha_1 \wedge \ldots \wedge \alpha_{l-1} \wedge \omega \wedge (d\omega)^{k-l+1} \equiv 0$$

Define J_l to be the set of all 1-forms α such that

$$\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_{l-1} \wedge \omega \wedge (d\omega)^{k-l} \equiv 0$$

Then:

(i) J_l is spanned by $2k-l$ 1-forms $\tau_1, \ldots, \tau_{2k-l}$.

(ii) If Φ is a 2-form satisfying

$$\Phi \wedge \alpha_1 \wedge \ldots \wedge \alpha_{l-1} \wedge \omega \wedge (d\omega)^{k-l} \equiv 0$$

then there exist 1-forms μ_i such that

$$\Phi = \sum_{i=1}^{2k-l} \mu_i \wedge \tau_i$$

Remark 3.1. [14] Let $\mathcal{I}$ be a subset of the space of all 1-forms α defined by:

$$\mathcal{I} = \{\alpha | \alpha \wedge \omega \wedge (d\omega)^{k-1} = 0\}$$

Then, it generates a differential ideal.

Proof. We claim that $\mathcal{I}$ generates a differential ideal. This is equivalent to the Forbenius condition: if $\alpha_1, \alpha_2, \ldots, \alpha_{2k-1}$ span $\mathcal{I}$, then there are 1-forms μ_{ij} such that

$$d\alpha_i = \sum_{j=1}^{2k-1} \mu_{ij} \wedge \alpha_j, \qquad \forall 1 \leq i \leq 2k-1. \tag{3.12}$$

To verify the equation (3.12), let a 1-form α belong to $\mathcal{I}$, then

$$\alpha \wedge \omega \wedge (d\omega)^{k-1} = 0. \tag{3.13}$$

We apply the exterior derivative to equation (3.13), we get

$$d\alpha \wedge \omega \wedge (d\omega)^{k-1} - \alpha \wedge (d\omega)^k = 0$$

By lemma 3.3, $\alpha \wedge (d\omega)^k = 0$. So,

$$d\alpha \wedge \omega \wedge (d\omega)^{k-1} = \alpha \wedge (d\omega)^k = 0.$$

By lemma 3.4 (ii), we obtain equation (3.12). □

Now we are ready to prove the sufficient condition: Assume the Ekeland-Nirenberg condition is satisfied at the origin. Without loss of generality, we assume that at the origin, $\omega(0) = dx^1$, and the subspace V is spanned by $dx^1, dx^2, \ldots, dx^k$. Thus, $N = V^\perp$ consists of all tangent vectors ξ, at the origin, such that

$$\xi^1 = \xi^2 = \ldots = \xi^k = 0$$

The symmetry of $\omega_{i,j}(0)$ on the subspace N implies that:

$$\omega_{i,j}(0) = \omega_{j,i}(0), \qquad \forall i, j > k$$

Thus,

$$d\omega(0) = dx^1 \wedge \alpha_1 + \tau$$

with $\tau = dx^2 \wedge \alpha_2 + \ldots + dx^k \wedge \alpha_k$, where each α_i involves only the dx^j with $j > i$, and so again at the origin:

$$\omega \wedge (d\omega)^{k-l} = \omega \wedge \tau^{k-l}, \text{ with } \tau^k = 0.$$

We need the following lemma in the proof:

Lemma 3.5. *[14] At the origin, if $\beta_1, \ldots, \beta_l$ are any l linear forms in V, then*

$$\beta_1 \wedge \ldots \wedge \beta_l \wedge \omega \wedge (d\omega)^{k-l} = 0.$$

Proof. Since $\beta_1, \ldots, \beta_l$ are 1-forms in V, then for all $i = 1, 2, \ldots, l$

$$\beta_i = \sum_{j=1}^{k} \beta_{ij} dx^j = \beta_{i1} dx^1 + \beta_i'$$

So,

$$\begin{aligned} \beta_1 \wedge \ldots \wedge \beta_l \wedge \omega \wedge (d\omega)^{k-l} &= \beta_1' \wedge \ldots \wedge \beta_l' \wedge \omega \wedge (d\omega)^{k-l} \\ &= \beta_1' \wedge \ldots \wedge \beta_l' \wedge \omega \wedge (\tau)^{k-l}. \end{aligned}$$

By the definition of τ, we know that its compontents involve $(k-l)$ products of $dx^2, \ldots, dx^k$ and each compontent in $\beta_1' \wedge \ldots \wedge \beta_l'$ involves l products of $dx^2, \ldots, dx^k$.

Thus, each compontent in the product of the two involves k 1-forms of $dx^2, \ldots, dx^k$, and hence each compontent is equal to zero. □

We are ready to start constructing $g_1, g_2, \ldots, g_k$.

Construction of g_1:

Define a subset $\mathcal{I}$ of the space of all 1-forms α by:

$$\mathcal{I} = \{\alpha | \alpha \wedge \omega \wedge (d\omega)^{k-1} = 0\}$$

Since $\mathcal{I}$ has dimension $2k-1$ and satisfies the Forbenius condition, by Forbenuis Theorem; there exist $2k-1$ functions $u_1, u_2, \ldots, u_{2k-1}$, the differentials of which span $\mathcal{I}$. We may choose $u_1, u_2, \ldots, u_k$ such that, at the origin:

$$du_i(0) = dx^i, \quad \forall i = 1, \ldots, k$$

$$u_j(0) = 0, \quad \forall j.$$

Since $\omega \in \mathcal{I}$, we may write:

$$\omega = \sum_{l=1}^{2k-1} a^l du_l. \tag{3.14}$$

with $a^1(0) = 1$ and $a^l(0) = 0$ for all $l > 1$. We will prove now that g_1 is strictly convex. So,

$$\omega_i = \sum_{l=1}^{2k-1} a^l \frac{\partial u_l}{\partial x^i}$$

and

$$\begin{aligned} \omega_{i,j}(0) &= \sum_{l=1}^{2k-1} \frac{\partial a^l}{\partial x^j}(0)\frac{\partial u_l}{\partial x^i}(0) + \sum_{l=1}^{2k-1} a^l(0)\frac{\partial^2 u_l}{\partial x^i \partial x^j}(0) \\ &= \sum_{l=1}^{2k-1} \frac{\partial a^l}{\partial x^j}(0)\frac{\partial u_l}{\partial x^i}(0) + \frac{\partial^2 u_1}{\partial x^i \partial x^j}(0). \end{aligned}$$

Since the Ekeland-Nirenberg condition is satisfied at the origin, then $\omega_{i,j}(0)$ is positive definite on N. But $\mathcal{I}(0)^\perp \subset V^\perp = N$, then $\omega_{i,j}(0)$ is positive definite on $\mathcal{I}(0)^\perp$. So, for each $\xi \in \mathcal{I}^\perp(0)$, we have

$$\sum_{i=1}^{n} \frac{\partial u_l(0)}{\partial x^i}\xi^i = 0, \qquad l = l, \ldots, 2k-1.$$

Then,

$$\begin{aligned}\sum_{i,j=1}^{n}\omega_{i,j}(0)\xi^i\xi^j &= \sum_{l=1}^{2k-1}\sum_{i,j=1}^{n}\frac{\partial a^l(0)}{\partial x^j}\frac{\partial u_l(0)}{\partial x^i}\xi^i\xi^j+\sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i\partial x^j}\xi^i\xi^j\\ &= \sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i\partial x^j}\xi^i\xi^j.\end{aligned}$$

We claim that there exists $c > 0$, such that

$$c\|\xi\|^2 \le \sum_{i,j=1}^{n}\omega_{i,j}(0)\xi^i\xi^j = \sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i\partial x^j}\xi^i\xi^j, \qquad \forall\xi\in\mathcal{I}^{\perp}(0)$$

we will prove the existence of the real number $c > 0$. Note that positive definiteness of $\omega_{i,j}(0)$ means:

$$\sum_{i,j=1}^{n}\omega_{i,j}(0)\xi^i\xi^j > 0, \quad \text{on} \quad N$$

.

Define E to be the unit sphere in N, and consider the following function

$$h : E \to R$$
$$\xi \to \sum_{i,j=1}^{n}\omega_{i,j}(0)\xi^i\xi^j.$$

h is a continuous, positive function and E is a compact set, then the minimum of this function is achieved and it is a positive number, which we call c. Thus, for each $\xi \in N$ and $\xi/\|\xi\| \in E$

$$c \le \sum_{i,j=1}^{n}\omega_{i,j}(0)\frac{\xi^i}{\|\xi\|}\frac{\xi^j}{\|\xi\|}.$$

□

Set

$$g_1 = u_1 + \epsilon_1 u_2 + K \sum_{l=1}^{2k-1} u_l^2$$

with $\epsilon_1 > 0$ small and K large.

It remains to show that g_1 satisfies the desired properties.

i. Since dg_1 is a combination of $du_1, \ldots, du_{2k-1}$, then $dg_1 \in \mathcal{I}$. Thus

$$dg_1 \wedge \omega \wedge (d\omega)^{k-1} = 0.$$

ii. The g_1 is strictly convex function at the origin. Since at the origin,

$$\sum_{i,j=1}^{n} \frac{\partial^2 g_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j = \sum_{i,j=1}^{n} \frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j + \epsilon_1 \sum_{i,j=1}^{n} \frac{\partial^2 u_2(0)}{\partial x^i \partial x^j}\xi^i\xi^j + 2K \sum_{l=1}^{2k-1}\sum_{i=1}^{n}\left(\frac{\partial u_l(0)}{\partial x^i}\xi^i\right)^2.$$

We consider two cases:

(a) For $\xi \in \mathcal{I}^{\perp}(0)$, we have

$$\sum_{i,j=1}^{n} \frac{\partial^2 g_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j = \sum_{i,j=1}^{n} \frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j + \epsilon_1 \sum_{i,j=1}^{n} \frac{\partial^2 u_2(0)}{\partial x^i \partial x^j}\xi^i\xi^j \geq \frac{c}{2}\|\xi\|^2$$

for small ϵ_1.

(b) For ξ belongs to complementary subspace of $\mathcal{I}^{\perp}(0)$, then

$$\sum_{i,j=1}^{n} \frac{\partial^2 g_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j = \sum_{i,j=1}^{n} \frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j + \epsilon_1 \sum_{i,j=1}^{n} \frac{\partial^2 u_2(0)}{\partial x^i \partial x^j}\xi^i\xi^j + 2K \sum_{l=1}^{2k-1}\sum_{i=1}^{n}\left(\frac{\partial u_l(0)}{\partial x^i}\xi^i\right)^2$$

$$\geq \frac{c}{2}||\xi||^2, c > 0. \qquad \text{For } K \text{ large enough.}$$

Finally,

$$dg_1 = du_1 + \epsilon_1 du_2 + 2K \sum_{l=1}^{2k-1} u_l du_l$$

At the origin,

$$dg_1(0) = dx^1 + \epsilon_1 dx^2.$$

Construction of g_l, for $1 \leq l \leq k-1$: We now argue by induction. Suppose we have constructed the functions

$$g_1, g_2, \dots, g_{l-1}$$

for $l \leq k-1$, and positive numbers

$$\epsilon_1, \epsilon_2, \dots, \epsilon_{l-1}$$

satisfying recursively,

$$\begin{aligned} dg_1 \wedge \omega \wedge (d\omega)^{k-1} &\equiv 0, \\ dg_2 \wedge dg_1 \wedge \omega \wedge (d\omega)^{k-2} &\equiv 0, \\ &\vdots \\ dg_{l-1} \wedge dg_{l-2} \wedge \dots \wedge dg_2 \wedge dg_1 \wedge \omega \wedge (d\omega)^{k-l+1} &\equiv 0. \end{aligned}$$

and at the origin,

$$dg_i(0) = dx^1 - \epsilon_i dx^i + \epsilon_i dx^{i+1}$$

for all $i = 2, 3, \dots, l-1$, while for $i = 1$,

$$dg_1(0) = dx^1 + \epsilon_1 dx^2.$$

Now, we construct g_l with similar properties. Define a subset $\mathcal{I}_l$ of the space of all 1-forms α by:

$$\mathcal{I}_l = \{\alpha | \alpha \wedge dg_{l-1} \wedge \dots \wedge d_1 \wedge \omega \wedge (d\omega)^{k-l} = 0\}$$

Since $\mathcal{I}_l$ generates a differential ideal, and has dimension $2k-l$, using Forbenius theorem, there exist $2k-l$ functions $u_1, u_2, \dots, u_{2k-l}$ spanning $\mathcal{I}_l$. From the definiton of dg_i at the origin, then $dg_i \in V$ for all $i = 1, 2, \dots, l-1$.

Let α be a 1-form in V, then by lemma 3.5 we get at the origin

$$\alpha \wedge dg_1 \wedge dg_2 \wedge \ldots \wedge dg_{l-1} \wedge \omega \wedge (d\omega)^{k-l} = 0$$

that means $\alpha \in \mathcal{I}_l(0)$. Thus, $V \subset \mathcal{I}_l(0)$. We may choose $u_1, u_2, \ldots, u_k$ such that, at the origin:

$$du_i(0) = dx^i, \quad \forall i = 1, \ldots, k$$

$$u_j(0) = 0, \quad \forall j.$$

Again,

$$\omega = \sum_{l=1}^{2k-l} a^l du_l. \tag{3.15}$$

with $a^1(0) = 1$ and $a^l(0) = 0$ for all $l > 1$. So,

$$\omega_i = \sum_{l=1}^{2k-l} a^l \frac{\partial u_l}{\partial x^i}$$

and

$$\begin{aligned}
\omega_{i,j}(0) &= \sum_{l=1}^{2k-l} \frac{\partial a^l}{\partial x^j}(0)\frac{\partial u_l}{\partial x^i}(0) + \sum_{l=1}^{2k-l} a^l(0)\frac{\partial^2 u_l}{\partial x^i \partial x^j}(0) \\
&= \sum_{l=1}^{2k-l} \frac{\partial a^l}{\partial x^j}(0)\frac{\partial u_l}{\partial x^i}(0) + \frac{\partial^2 u_1}{\partial x^i \partial x^j}(0).
\end{aligned}$$

By assumption, $\omega_{i,j}(0)$ is positive definite on N. But $\mathcal{I}_l(0)^\perp \subset V^\perp = N$, then $\omega_{i,j}(0)$ is positive definite on $\mathcal{I}(0)^\perp$. So, for each $\xi \in \mathcal{I}_l^\perp(0)$, we have

$$\sum_{i=1}^{n} \frac{\partial u_l(0)}{\partial x^i}\xi^i = 0, \qquad l = l, \ldots, 2k - l.$$

Then,

$$\begin{aligned}\sum_{i,j=1}^{n} \omega_{i,j}(0)\xi^i\xi^j &= \sum_{l=1}^{2k-l}\sum_{i,j=1}^{n} \frac{\partial a^l(0)}{\partial x^j}\frac{\partial u_l(0)}{\partial x^i}\xi^i\xi^j + \sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j \\ &= \sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j.\end{aligned}$$

By our assumption on V, it follows that, for some $c > 0$, we have

$$\sum_{i,j=1}^{n} \omega_{i,j}(0)\xi^i\xi^j = \sum_{i,j=1}^{n}\frac{\partial^2 u_1(0)}{\partial x^i \partial x^j}\xi^i\xi^j \geq c||\xi||^2, \qquad \forall \xi \in I_l(0)^{\perp}.$$

We now define

$$g_l = u_1 - \epsilon_l u_l + \epsilon_l u_{l+1} + K\sum_{l=1}^{2k-l}(u_l)^2.$$

i. Since dg_l is a combination of $du_1, \ldots, du_{2k-l}$, then $dg_l \in \mathcal{I}_l$. Thus

$$dg_l \wedge dg_{l-1} \wedge \ldots \wedge dg_1 \wedge \omega \wedge (d\omega)^{k-l} = 0.$$

ii. With similar discussion as before, the g_l is strictly convex function at the origin. Since at the origin, we find that

$$\sum_{i,j=1}^{n}\frac{\partial^2 g_l(0)}{\partial x^i \partial x^j}\xi^i\xi^j \geq \frac{c}{2}||\xi||^2$$

Hence, at the origin

$$dg_l(0) = dx^1 - \epsilon_l dx^l + \epsilon_l dx^{l+1}.$$

Construction of g_k:

We have constructed $g_1, g_2, \ldots, g_{k-1}$ with convex property, finally we construct g_k. Define a subset $\mathcal{I}_k$ of the space of all 1-forms α by:

$$\mathcal{I}_k = \{\alpha | \alpha \wedge dg_{k-1} \wedge \ldots \wedge d_1 \wedge \omega = 0\}$$

Since $\mathcal{I}_k$ generates a differential ideal, and has dimension k, by Forbenuis Theorem; there exist k functions $w_1, w_2, \ldots, w_k$, the differentials of which span $\mathcal{I}_k$. We may choose $w_1, w_2, \ldots, w_k$ such that, at the origin:

$$dw_i(0) = dx^i, \quad \forall i = 1, \ldots, k$$

$$w_j(0) = 0, \quad \forall j.$$

Since $\omega \in \mathcal{I}_k$, we may write:

$$\omega = \sum_{l=1}^{k} a^l dw_l \tag{3.16}$$

with $a^1(0) = 1$ and $a^l(0) = 0$ for all $l > 1$.

We now set

$$g_k = w_1 - \epsilon_k w_k + K \sum_{l=1}^{k} (w_l)^2.$$

As before, by assumption, $\omega_{i,j}(0)$ is positive definite on $N = V^\perp = \mathcal{I}_k^\perp$. Then for small $\epsilon_k > 0$ and large K, we get

$$\sum_{i,j=1}^{n} \frac{\partial^2 g_k(0)}{\partial x^i \partial x^j} \xi^i \xi^j \geq \frac{c}{2} ||\xi||^2.$$

Hence, at the origin

$$dg_k(0) = dx^1 - \epsilon_k dx^k.$$

To complete the proof of the convex Darboux theorem, we must show that in the repersentation

$$\omega = \sum_{l=1}^{k} f^l dg_l$$

all the f^l are positive functions at the origin.

Remark 3.2. At the origin, the $dg_l(0)$ are independent, so the $f^l(0)$ are unique.

But at the origin,

$$\begin{aligned}
\omega(0) &= dx^1 \\
dg_1(0) &= dx^1 + \epsilon_1 dx^2 \\
dg_2(0) &= dx^1 - \epsilon_2 dx^2 + \epsilon_2 dx^3 \\
&\vdots \\
dg_{k-1}(0) &= dx^1 - \epsilon_{k-1} dx^{k-1} + \epsilon_{k-1} dx^k \\
dg_k(0) &= dx^1 - \epsilon_k dx^k.
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{1}{\epsilon_1} dg_1(0) &= \frac{1}{\epsilon_1} dx^1 + dx^2 \\
\frac{1}{\epsilon_2} dg_2(0) &= \frac{1}{\epsilon_2} dx^1 - dx^2 + dx^3 \\
&\cdots \\
\frac{1}{\epsilon_{k-1}} dg_{k-1}(0) &= \frac{1}{\epsilon_{k-1}} dx^1 - dx^{k-1} + dx^k \\
\frac{1}{\epsilon_k} dg_k(0) &= \frac{1}{\epsilon_k} dx^1 - \epsilon_k dx^k.
\end{aligned}$$

Summing up, we get

$$\sum_{i=1}^{k} \frac{1}{\epsilon_i} dg_i(0) = (\sum_{i=1}^{k} \frac{1}{\epsilon_i}) dx^1$$

which gives the desired decompostion

$$\omega(0) = \sum_{l=1}^{k} f^l(0) dg_l(0)$$

with

$$f^l(0) = \frac{1}{\epsilon_l \sum_{i=1}^k \epsilon_i} > 0.$$

This concludes the proof.

Chapter 4

Decomposition Of Homogeneous Differential Forms

Many economic functions are homogeneous of different degrees. For example, the demand function is homogeneous of degree zero when the income function is homogeneous of degree one. This property is called in economics " the absence of money illusion", which means that if you multiply prices and income by same constant then the consumer is indifferent. In this chapter, we ask the following question: what are the necessary and sufficient conditions for a given k-homogeneous differential 1-form ω to be decomposed as

$$\omega = \sum_{i=1}^{l} a^i(x) du_i(x)$$

where the functions $a^i(x)$ are $(k+1)$-homogeneous and the functions $u_i(x)$ are 0-homogeneous?

4.1 Integrability Of Homogeneous Differential Forms

Consider the vector space $\mathbb{R}^n$ with coordinate system $(x^1, x^2, \ldots, x^n)$. Define the vector field X in the tangent space of $\mathbb{R}^n$; that is, $X \in T\mathbb{R}^n$ by

$$X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

Define the differential 1-form ω by

$$\omega = \sum_{i}^{n} \omega_i(x) dx^i$$

The exterior derivative of ω, denoted by $d\omega$, is the 2-form

$$d\omega = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i$$

In the following theorem we study the simplest case in which we find an L-homogeneous function $g(x)$ such that $\omega = dg(x)$.

Theorem 4.1. *Let ω be an $(L-1)$-homogeneous differential 1-form such that $L \neq 0$. Then, there exists an L-homogeneous function $g(x)$ such that $\omega = dg(x)$ if and only if $d\omega = 0$. Moreover, $g(x) = \frac{\iota_X \omega}{L}$.*

Proof. Suppose that a given 1-form ω is homogeneous of degree $L-1$ and $d\omega = 0$. Using Lie derivative, then,

$$\mathcal{L}_X \omega = L\omega$$

On the other hand, $\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega$. Since $d\omega = 0$, then

$$\mathcal{L}_X \omega = d\iota_X \omega = L\omega$$

So,

$$\omega = \frac{1}{L} d\iota_X \omega$$

Since $d\omega = 0$, by Poincaré lemma, there exists a function g such that $\omega = dg$. So we have

$$\omega = d(\frac{1}{L}\iota_X \omega) = dg$$

Then, $g = \frac{1}{L}\iota_X \omega$. Moreover, the function g is L-homogeneous since $\iota_X dg = \iota_X \omega = Lg(x)$. Hence, we get the required result. $\square$

Corollary 4.2. *Let ω be a C^2, -1-homogeneous differential 1-form. Then, there exists a 0-homogeneous function $g(x)$ such that $\omega = dg(x)$ if and only if $\iota_X \omega = 0$ and $d\omega = 0$.*

Theorem 4.3. *Let ω be a C^1, k-homogeneous differential m-form such that $m+k \neq 0$. Then, $d\omega = 0$ if and only if there exists a differential $(m-1)$-form σ such that $\omega = d\sigma$, where σ is given by*

$$\sigma = \frac{\iota_X \omega}{k+m}.$$

Proof. Suppose that a given m-form ω is homogeneous of degree k and $d\omega = 0$. Using Lie derivative. Then,

$$\mathcal{L}_X \omega = (k+m)\omega$$

On the other hand, $\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega$. Since $d\omega = 0$, then

$$\mathcal{L}_X \omega = d\iota_X \omega = (k+m)\omega$$

So,

$$\omega = \frac{1}{k+m} d\iota_X \omega.$$

Since $d\omega = 0$, by Poincaré lemma, there exists a differential $(m-1)$-form σ

such that $\omega = d\sigma$. So we have

$$\omega = d(\frac{1}{k+m}\iota_X\omega) = d\sigma$$

Then, we get an $(m-1)$-form σ such that $\omega = d\sigma$, where σ is given by

$$\sigma = \frac{\iota_X\omega}{k+m}$$

This completes the proof. □

We now consider the following lemma.

Lemma 4.4. *Let ω be a differential 1-form such that $d\omega = 0$. Then, ω is -1-homogeneous if and only if $d(\iota_X\omega) = 0$.*

Proof. Let ω be a differential 1-form such that $d\omega = 0$. Then, ω is -1-homogeneous if and only if $\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega) = 0$. Hence, ω is -1-homogeneous if and only if $d(\iota_X\omega) = 0$ □

In the following we provide some examples.

Example 4.1. Consider the vector space $\mathbb{R}^2$ with coordinate system (x, y) and the differential 1-form

$$\omega = \frac{-2y^2}{x^3}dx + \frac{2y}{x^2}dy$$

So, ω is -1-homogeneous and

$$\begin{aligned} d\omega &= \frac{-4y}{x^3}dy \wedge dx + \frac{-4y}{x^3}dx \wedge dy \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} \iota_X\omega &= \frac{-2y^2x}{x^3} + \frac{2y^2}{x^2} \\ &= 0 \end{aligned}$$

where $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Then, there exists a 0-homogeneous function $g = \frac{y^2}{x^2}$ such that $\omega = dg$.

Example 4.2. Consider the vector space $\mathbb{R}^2$ with coordinate system (x, y) and the differential 1-form

$$\omega = 3(x-y)^2 dx - 3(x-y)^2 dy.$$

So, ω is 2-homogeneous and

$$\begin{aligned} d\omega &= -6(x-y)dy \wedge dx - 6(x-y)dx \wedge dy \\ &= 0. \end{aligned}$$

So, ω is closed on $\mathbb{R}^2$. By Poincaré lemma, it is exact. Then, $\omega = dg$ where $g(x) = \frac{1}{3}\iota_X\omega = x^3 - 3x^2y + 3xy^2 - y^3$.

In the next theorem we answer the following question: given a homogeneous differential 1-form ω, do there exist homogeneous functions $f(x)$ and $g(x)$ such that $\omega = f(x)dg(x)$?

Theorem 4.5. *[5] Let ω be a k-homogeneous differential 1-form such that $\omega \wedge d\omega = 0$ and $\iota_X\omega \neq 0$. Then, there exists a function g such that $\omega = (\iota_X\omega)dg$.*

Proof. Suppose that a given 1-form ω is homogeneous of degree k and $\omega \wedge d\omega = 0$. We know that $\mathcal{L}_X\omega = \iota_X d\omega + d\iota_X\omega$. Then,

$$\omega \wedge \mathcal{L}_X\omega = \omega \wedge (\iota_X d\omega + d(\iota_X\omega)) = \omega \wedge (\iota_X d\omega) + \omega \wedge d(\iota_X\omega) \tag{4.1}$$

Since

$$\iota_X(\omega \wedge d\omega) = (\iota_X\omega)d\omega - \omega \wedge (\iota_X d\omega)$$

Substitute the value of $\omega \wedge (\iota_X d\omega)$ into equation (4.1), we get

$$\omega \wedge \mathcal{L}_X\omega = -\iota_X(\omega \wedge d\omega) + (\iota_X\omega)d\omega + \omega \wedge d(\iota_X\omega)$$

Since $\omega \wedge d\omega = 0$ and ω is k-homogeneous then $\omega \wedge \mathcal{L}_X\omega = 0$. Then, the last equation implies that

$$d((\iota_X\omega)^{-1}\omega) = 0$$

By Poincaré lemma, there exists a function g such that $\omega = (\iota_X\omega)dg$. The proof is complete. □

Example 4.3. [5] Consider the vector space $\mathbb{R}^3$ with coordinate system (x, y, z) and the differential 1-form

$$\omega = 2z(y+z)dx - 2xzdy + ((y+z)^2 - x^2 - 2xz)dz.$$

Then, we notice that ω is 2-homogeneous and

$$\omega \wedge d\omega = 0.$$

Then,

$$\iota_X\omega = z((y+z)^2 - x^2)$$

where $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Then,

$$\omega = (\iota_X\omega)dg = (\iota_X\omega)d\ln\left|\frac{z(x+y+z)}{-x+y+z}\right|.$$

Notice that $g(x, y, z)$ is non-homogeneous function.

Lemma 4.6. *[5] If ω is a C^1, k-homogeneous differential 1-form such that $\iota_X\omega = 0$ then $\iota_X d\omega = (k+1)\omega$*

Proof. ω is k-homogeneous if and only if $\mathcal{L}_X\omega = (k+1)\omega$. But, we know that

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega$$

Since $\iota_X\omega = 0$. Then, we get

$$\mathcal{L}_X\omega = \iota_X d\omega = (k+1)\omega$$

The proof is complete. □

Proposition 4.7. *[5] Let ω be a C^1 differential 1-form such that $\iota_X\omega = 0$. Then, $\omega\wedge d\omega = 0$ with ω is k-homogeneous if and only if there is a differential 1-form β such that $d\omega = \beta\wedge\omega$ with $\iota_X\beta = k+1$*

Proof. Let ω be a C^1 differential 1-form such that $\iota_X\omega = 0$. If $d\omega = \beta\wedge\omega$ then $\omega\wedge d\omega = 0$. We take the interior product of both sides of $d\omega = \beta\wedge\omega$, then

$$\iota_X d\omega = (\iota_X\beta)\omega - (\iota_X\omega)\beta = (k+1)\omega$$

So

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega = (k+1)\omega$$

Then, ω is k-homogeneous. Conversely, if $\omega\wedge d\omega = 0$ with ω is k-homogeneous then there exists a differential 1-form β such that $d\omega = \beta\wedge\omega$. Moreover,

$$\iota_X d\omega = (\iota_X\beta)\omega - (\iota_X\omega)\beta = (k+1)\omega.$$

So, $\iota_X\beta = k+1$. Hence, we get the required result.

□

Theorem 4.8. *[5] Let ω be a C^1, k-homogeneous differential 1-form such that $\iota_X\omega = 0$ and $\omega\wedge d\omega = 0$ in a neighborhood $\mathcal{U}$ of some point $\bar{x}$. Then, there exist a $(k+1)$-homogeneous function f and a 0-homogeneous function g, defiend in a neighborhood; $\mathcal{V}\subset\mathcal{U}$, such that $\omega(x) = f(x)dg(x)$.*

Proof. Suppose that $\omega \wedge d\omega = 0$. By Darboux theorem, there exist two functions f and g such that $\omega(x) = f(x)dg(x)$. Since $\iota_X\omega = 0$, then

$$\iota_X dg = 0.$$

That is; g is a 0-homogeneous function. We have

$$d\omega = df \wedge dg \quad \text{and} \quad dg = \frac{\omega}{f}.$$

It follows that

$$d\omega = \frac{df}{f} \wedge \omega.$$

Apply the vector field $X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ to both sides of previous equation and use lemma(4.6), we get

$$(k+1)\omega = (\iota_X \frac{df}{f})\omega$$

Thus, $\iota_X df = (k+1)f$, which means that $f(x)$ is $(k+1)$-homogeneous function. This completes the proof. □

Theorem 4.9. *[5] Let $\omega(x)$ be a C^1, k-homogeneous differential 1-form. Suppose that ω has rank $r = 2k$ in a neighborhood; $\mathcal{U}$, of some point $\bar{x}$ and $\iota_X\omega = 0$ for all $x \in \mathcal{U}$. Then, there exist $2k$ functions, $a^1, \ldots, a^k, u_1, \ldots, u_k$ defined in a neighborhood, $\mathcal{V} \subset \mathcal{U}$, such that*

(a) $\omega(x) = \sum_{i}^{k} a^i(x)du_i(x)$.

(b) The functions $a^1, \ldots, a^k$ are $(k+1)$-homogeneous and $u_1, \ldots, u_k$ are 0-homogeneous.

Proof. $\omega(x)$ has rank $2k$ in a neighborhood $\mathcal{U}$; that is,

$$\omega \wedge (d\omega)^{k-1} \neq 0, \quad \omega \wedge (d\omega)^k = 0.$$

Then, part (a) follows by Darboux theorem. To Prove part (b), we take the exterior derivative of both sides of $\omega(x) = \sum_i^k a^i(x)du_i(x)$. We get the following expression for $d\omega$:

$$d\omega = \sum_{i=1}^{k} da^i \wedge du_i \tag{4.2}$$

Assume, without loss of generality, that $a^1(x) \neq 0$ for all $x \in \mathcal{V}$. Then

$$du_1 = \frac{1}{a^1}(\omega - \sum_{i=2}^{l} a^i du_i)$$

Substituite the value of du_1 in the equation (4.2), we get

$$d\omega = \frac{da^1}{a^1} \wedge \omega + \sum_{i=2}^{k} \left(da^i - \frac{da^1}{a^1} a^i \right) \wedge du_i$$

Applying the vector field $X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ to both sides of this equation and using lemma(4.6), we get

$$(k+1)\omega = \iota_X \frac{da^1}{a^1} \omega + \sum_{i=2}^{k} \iota_X \left(da^i - \frac{da^1}{a^1} a^i \right) du_i - \sum_{i=2}^{k} \left(da^i - \frac{da^1}{a^1} a^i \right) \iota_X du_i.$$

Substituite for ω from part (a) into the previous equation, we get

$$\left(\iota_X \frac{da^1}{a^1} - (k+1) \right) \sum_i^k a^i(x) du_i(x) + \sum_{i=2}^{k} \iota_X \left(da^i - \frac{da^1}{a^1} a^i \right) du_i - \sum_{i=2}^{k} \left(da^i - \frac{da^1}{a^1} a^i \right) \iota_X du_i = 0.$$

Rearranging terms, we get

$$\begin{aligned} \left(\iota_X \frac{da^1}{a^1(x)} - (k+1) \right) a^1(x) du_1 \quad &+ \quad \sum_{i=2}^{k} \left[\iota_X \left(da^i - \frac{da^1}{a^1(x)} a^i(x) \right) + \left(\iota_X \frac{da^1}{a^1(x)} - (k+1) \right) a^i(x) \right] du_i \\ &- \quad \sum_{i=2}^{k} (\iota_X du_i) da^i + \frac{1}{a^1(x)} \sum_{i=2}^{k} a^i(x) (\iota_X du_i) da^1 = 0 \end{aligned}$$

Since $da^1, \ldots, da^l, du_1, \ldots, du_l$ are linearly independent differential 1-forms, we find that

$$\iota_X du_i(x) = 0 \quad \text{and} \quad \iota_X da^i(x) = (k+1) a^i(x)$$

which means that the function $u_i(x)$ is 0-homogeneous and the function $a^i(x)$ is $(k+1)$-homogeneous for all $i = 1, 2, \ldots, k$. The proof is complete. □

More generally, using Darboux theorem and the previous result, we obtain the following theorem.

Theorem 4.10. *[5] Let $\omega(x)$ be a C^1, k-homogeneous differential 1-form. The following statments are equivalent:*

i. ω has rank $r = 2l$ in a neighborhood, $\mathcal{U}$, of some point $\bar{x}$ and $\iota_X\omega = 0$ for all $x \in \mathcal{U}$.

ii. There exist $2l-1$ linearly independent 1-forms $\gamma, \alpha_1, \alpha_1, \ldots, \alpha_{l-1}$, $\beta^1, \beta^2, \ldots, \beta^{l-1}$ such that

$$d\omega = \omega \wedge \gamma + \sum_{i=1}^{k-1} \alpha_i \wedge \beta^i$$

with $\iota_X\gamma = k+1, \iota_X\alpha_i = 0$ and $\iota_X\beta^i = 0$ for all $i = 1, \ldots, l-1$ in U.

iii. There exist $2l$ functions, $a^1, \ldots, a^l, u_1, \ldots, u_l$ defined in a neighborhood, $\mathcal{V} \subset \mathcal{U}$, such that

$$\omega = \sum_{i}^{l} a^i(x) du_i(x)$$

where $a^1, \ldots, a^l$ are $k+1$-homogeneous and $u_1, \ldots, u_l$ are 0-homogeneous.

Proof. (a) implies (b). Since $(d\omega)^l \neq 0$ and $\omega \wedge (d\omega)^l = 0$ then there exist $2l-1$ differential 1-forms $\gamma, \alpha_1, \alpha_1, \ldots, \alpha_{l-1}, \beta^1, \beta^2, \ldots, \beta^{l-1}$ such that

$$d\omega = \omega \wedge \gamma + \sum_{i=1}^{l-1} \alpha_i \wedge \beta^i$$

We notice that

$$(d\omega)^l = l!(\alpha_1 \wedge \beta^1) \wedge \ldots \wedge (\alpha_{l-1} \wedge \beta^{l-1}) \wedge \omega \wedge \gamma \neq 0.$$

It follows that the 1-form $\alpha_1, \ldots, \alpha_{l-1}, \beta^1, \ldots, \beta^{l-1}, \omega, \gamma$ are linearly independent. Then,

$$\iota_X d\omega = (\iota_X \gamma)\omega - (\iota_X \omega)\gamma + \sum_{i=1}^{l-1} \left((\iota_X \alpha_i)\beta^i - (\iota_X \beta^i)\alpha_i \right).$$

Using the fact that $\iota_X d\omega = (k+1)\omega$, $\iota_X \omega = 0$ and the linear independence of $\alpha_1, \ldots, \alpha_{l-1}, \beta^1, \ldots, \beta^{l-1}, \omega, \gamma$, it follows that

$$\iota_X \gamma = k+1, \iota_X \alpha_i = 0, \iota_X \beta^i = 0, \quad \forall i = 1, \ldots, l-1.$$

(a) implies (c). Conversely follow form Barboux theorem and the homogeneity of the functions $a_1(x), \ldots, a_l(x), u^1(x), \ldots, u^l(x)$ follows form theorem 4.9. This complete the proof. $\square$

4.2 Why Does Ekeland-Nirenberg Theorem Fail In The Homogeneous Setting?

Given a smooth differential 1-form defined on a neighborhood; $\mathcal{U}$, of some point $\bar{x}$ in $\mathbb{R}^n$

$$\omega = \sum_{i=1}^{n} \omega_i dx^i$$

Under what conditions can we represent a smooth 0-homogeneous differential 1-form ω in the form:

$$\omega = \sum_{l=1}^{k} f^l dg_l \tag{4.3}$$

on a neighborhood; $\mathcal{V} \subset \mathcal{U}$, of $\bar{x}$, where the f^l are 1-homogeneous positive functions and the g_l are 0-homogeneous convex (or quasiconvex) functions? This decomposition is encountered in many economic applications. For example, the problems of characterization of excess demand functions, Marshallian demand functions when consumers income function $w(\pi)$ is homogeneous of degree one and also for household demands in a similar setting. By a classic result in exterior differential calculus; Darboux Theorem, if ω has rank $2k$,

$$\omega \wedge d(\omega)^{k-1} \neq 0 \quad \text{and} \quad \omega \wedge (d\omega)^k = 0 \quad \text{on} \quad \mathcal{U}.$$

then (4.3) holds. If ω satisfies (4.3), then

$$d\omega = \sum_{l=1}^{k} df^l \wedge dg_l$$

and

$$(d\omega)^k = k! df^1 \wedge dg_1 \wedge df^2 \wedge dg_2 \ldots \wedge df^k \wedge dg_k$$

hence,

$$\omega \wedge (d\omega)^k = 0$$

Darboux theorem does not give any guarantee for positiveness of the coefficients and convexity of the coordinates. Moreover, the Ekeland and Nirenberg gave a necessary and sufficient condition for just the positivity of f^l and convexity of g_l. Define the vector field X as

$$X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

We denote by $\iota_X \omega$, the interior product between the vector field X and the differential 1-form ω. The Ekeland and Nirenberg introduce the subspace $\mathcal{I}$ defined by

$$\mathcal{I} = \{\alpha | \alpha \wedge \omega \wedge (d\omega)^{k-1} = 0\}$$

Ekeland-Nirenberg Condition: There is a k-dimensional subspace V of $\mathcal{I}(0)$, containing $\omega(0)$, such that on $N = V^{\perp}$, the matrix $\omega_{i,j}(0)$ is symmetric and positive definite.

As we mentioned in previous section, if the 0-homogeneous differential 1-form ω can be represented in the form

$$\omega = \sum_{l=1}^{k} f^l dg_l$$

where the f^l are 1-homogeneous functions and the g_l are 0-homogeneous functions. Then,

$$\begin{aligned}\iota_X\omega &= \sum_{l=1}^{k} f^l \frac{\partial g_l}{\partial x^1}x^1 + \sum_{l=1}^{k} f^l \frac{\partial g_l}{\partial x^2}x^2 + \ldots + \sum_{l=1}^{k} f^l \frac{\partial g_l}{\partial x^n}x^n \\ &= \sum_{i=1}^{n} f^1 \frac{\partial g_1}{\partial x^i}x^i + \sum_{i=1}^{n} f^2 \frac{\partial g_2}{\partial x^i}x^i + \ldots + \sum_{i=1}^{n} f^k \frac{\partial g_k}{\partial x^i}x^i\end{aligned}$$

The functions g_l are 0-homogeneous, Using Euler's formula we get

$$\sum_{i=1}^{n} f^l \frac{\partial g_l}{\partial x^i}x^i = f^l \sum_{i=1}^{n} \frac{\partial g_l}{\partial x^i}x^i = 0, \quad \forall l = 1, 2, \ldots, k.$$

So, $\iota_X\omega = 0$. Then, the subspace $\mathcal{I}$ is of dimension $2k-1$ and is spanned by

$$\mathcal{I}(x) = \{\omega, \alpha_1, \ldots, \alpha_{k-1}, \beta^1, \ldots, \beta^{k-1}\}$$

Clearly, $x \in \mathcal{I}^{\perp}(x)$ since $\iota_X\omega = 0, \iota_X\alpha_i = 0, \iota_X\beta^i = 0$ for all $i = 1, \ldots, k-1$. Therefore, **Ekeland-Nirenberg Condition** cannot be fulfilled in the homogeneous setting which is natural since the functions $g_l, l = 1, \ldots, k$ cannot be strictly convex.

$$N = V^{\perp} \quad \text{and} \quad V \subset \mathcal{I}(x) \implies \mathcal{I}^{\perp}(x) \subset V^{\perp}$$

Hence, The matrix $\omega_{i,j}$ cannot be positive definite or negative definite on $N = V^{\perp}$. In the previous section we proved the existence of homogeneous decomposition, but we need also the additional positivity and convexity conditions.

Chapter 5

Applications

Consider a household that consists of M members (consumers). Each consumer is characterized by his own utility function

$$U^1, U^2, \ldots, U^M : \mathbb{R}^{N(M+1)} \to \mathbb{R}.$$

So that $U^m(y_1, y_2, \ldots, y_M, Y)$ where $y_m \in \mathbb{R}^N_+$ is member m's private consumption and $Y \in \mathbb{R}^N_+$ is the household's common consumption of public goods and U^m is increasing and strongly concave.

We assume that the decision process within the household is Pareto efficient; that is,

Axiom 1. *The outcome of the household decision process is Pareto efficient; that is, for any price vector, the consumption vector* $(y_1, y_2, \ldots, y_M, Y)$ *chosen by the household is such that no other* $(\hat{y_1}, \hat{y_2}, \ldots, \hat{y_M}, \hat{Y})$ *in budget set could make all consumers better off with at least one of them in a strict sense.*

The set of Pareto efficient allocations can be characterized by maximizing a weighted sum of utility functions $\sum_{m=1}^{M} \mu_m(\pi) U^m(y_1, y_2, \ldots, y_M, Y)$, where the price-dependent functions $\mu_1 \geq 0, \mu_2 \geq 0, \ldots, \mu_M \geq 0$, are Pareto weights that satisfy the normalization condition $\sum\limits_{m=1}^{M} \mu_m(\pi) = 1$. The $\mu_m(\pi)$ represents the power of member m within the household.

The collective demand function is the solution to the utility maximization problem under the budget constraint $\pi^T \xi \leq w(\pi)$, where $\pi \in \mathbb{R}^N_{++}$ is the price vector and $w(\pi)$ is the collective income function, where π^T is the transpose of π.

The utility maximization problem under the budget constraint takes the form:

$$\max_x U(x, \mu) \quad \text{subject to} \quad \pi^T x \leq w(\pi)$$

where $U(x, \mu)$ is the utility function that takes the form:

$$U(x, \mu) = \max_{y_1, y_2, \ldots, y_M, Y} \left\{ \sum_{m=1}^{M} \mu_m(\pi) U^m(y_1, y_2, \ldots, y_M, Y) | y = x \right\}$$

where x is the total purchases of the household, $w(\pi)$ is the household's income function and

$$y = \sum_{m=1}^{M} y_m + Y$$

The solution of this problem is characterized in [11] when the income function is price dependent.

We define a differential 1-form and set up an integration problem. The integration problem splits into Mathematical integration problem and Economic integration problem.

- **Mathematical integration.** Given function $\xi(\pi) \in \mathbb{R}^N_+$, what are the necessary and sufficient conditions for the existance of $2M$ functions $\lambda_l(\pi), V^l(\pi), l = 1, \ldots, M$ that satisfy the equation

$$\sum_{l=1}^{M} \lambda_l(p) \frac{\partial V^l}{\partial \pi^i} = \xi^i - \frac{\partial w}{\partial \pi^i} \tag{5.1}$$

with $w(\pi) = \pi^T \xi$.

- **Economic integration.** In addition to mathematical integration, we impose the following condition on the functions that satisfy equation (5.1); the functions $\lambda_l(\pi)$ are positive and the functions $V^l(\pi)$ are strongly concave.

The necessary and sufficient conditions for mathematical integration will be solved using Darboux Theorem [15].

5.1 Collective Demand Function: Non-homogeneous Case

Consider a household that consists of M members. Each consumer is characterized by his own utility function

$$U^1, U^2, \ldots, U^M : \mathbb{R}^{N(M+1)} \to \mathbb{R}.$$

So that $U^m(y_1, y_2, \ldots, y_M, Y)$ where $y_m \in \mathbb{R}^N_+$ is member m's private consumption and $Y \in \mathbb{R}^N_+$ is the household's common consumption of public goods and U^m is increasing and strongly concave.

Pareto optimal allocations are characterized by the following maximization problem

$$(\mathcal{F}) \begin{cases} \max\limits_{y_1, y_2, \ldots, y_M, Y} \sum\limits_{m=1}^{M} \mu_m(\pi) U^m(y_1, y_2, \ldots, y_M, Y) \\ \text{subject to} \\ x = y \quad \text{and} \quad \pi^T x \le w(\pi). \end{cases}$$

where x is the total purchases of the household, $w(\pi)$ is the household's income function and

$$y = \sum_{m=1}^{M} y_m + Y$$

We assume the household's income function $w(\pi)$ is nonhomogeneous. The above maximization problem can be written as a two stage maximization problem

$$\max_x U(x,\mu) \quad \text{subject to} \quad \pi^T x \leq w(\pi) \tag{P1}$$

where

$$U(x,\mu) = \max_{y_1,y_2,\ldots,y_M,Y} \left\{ \sum_{m=1}^{M} \mu_m(\pi) U^m(y_1,y_2,\ldots,y_M,Y) | y = x \right\} \tag{P2}$$

we note that the solution $x = \xi(\pi)$ of problem (P1) is obsevable, whereas the solution $(y_1, y_2, \ldots, y_M, Y)$ of problem (P2) is not.
Define the function $\hat{V}$ as follow:

$$\hat{V}(\pi,\mu) = \max_x \{U(x,\mu)|\pi^T x \leq w(\pi)\}$$

Let $x = \hat{\xi}(\pi,\mu)$ be the maximizer that satisfies $\pi^T \hat{\xi}(\pi,\mu) = w(\pi)$ which is a Marshallian demand function when consumer's income is price dependent. The collective indirect utility function is defined as follow:

$$V(\pi) = \hat{V}(\pi,\mu(\pi)) = U(\xi(\pi,\mu(\pi)),\mu(\pi)) \tag{5.2}$$

Theorem 5.1. *Let $V(\pi)$ be the indirect utility function defined by (5.2). If $w(\pi)$ is a convex function then $V(\pi)$ is quasi-convex.*

Proof. Let $\bar{\pi}$ and $\hat{\pi}$ be price vectors. Consider the combinations

$$\tilde{\pi} = t\hat{\pi} + (1-t)\bar{\pi}, \quad \text{for} \quad t \in (0,1).$$

Suppose that $V(\bar{\pi}) \leq U(x,\mu)$ and $V(\hat{\pi}) \leq U(x,\mu)$. We want to prove that

$V(\tilde{\pi}) \leq \max\{V(\bar{\pi}), V(\hat{\pi})\}$. Introduce the following sets

$$\hat{S} = \{x|\hat{\pi}^T x \leq w(\hat{\pi})\}, \quad \bar{S} = \{x|\bar{\pi}^T x \leq w(\bar{\pi})\}, \quad \tilde{S} = \{x|\tilde{\pi}^T x \leq w(\tilde{\pi})\}.$$

We claim that $\tilde{S} \subset \bar{S} \cup \hat{S}$. If this is not the case, then there exists x such that $\bar{\pi}^T x > w(\bar{\pi})$ and $\hat{\pi}^T x > w(\hat{\pi})$ whereas $\tilde{\pi}^T x \leq w(\tilde{\pi})$. It follows that for any $t \in (0,1)$, $t\hat{\pi}^T x > tw(\hat{\pi})$ and $(1-t)\bar{\pi}^T x > (1-t)w(\bar{\pi})$. Adding up the last two inqualities and using the convexity of $w(\pi)$, we get

$$\tilde{\pi}^T x = (t\hat{\pi} + (1-t)\bar{\pi})^T x > tw(\bar{\pi}) + (1-t)w(\hat{\pi}) \geq w(t\hat{\pi} + (1-t)\bar{\pi}) = w(\tilde{\pi})$$

Hence, $\tilde{\pi}^T x \geq w(\tilde{\pi})$ which is a contradiction. So $\tilde{S} \subset \bar{S} \cup \hat{S}$ which implies that

$$V(\tilde{\pi}) = \max_{x \in \tilde{S}} U(x, \mu) \leq \max_{x \in \bar{S} \cup \hat{S}} U(x, \mu) = \max\{V(\bar{\pi}), V(\hat{\pi})\}$$

which means that $V(\pi)$ is quasi-convex. Hence, we get the required result.

□

The map $\pi \rightarrow \hat{\xi}(\pi, \mu)$ is the standard Marshalian demand function associated to $x \rightarrow U(x, \mu)$. This map satisfies the budget constraint $\pi^T \hat{\xi}(\pi, \mu_1, \ldots, \mu_M) = w(\pi)$ and the extended Slutsky matrix $S(\pi)$ defined by

$$S(\pi) = D_\pi \hat{\xi} - \frac{1}{\pi^T (D_\pi \hat{\xi}) \pi} (D_\pi \hat{\xi}) \pi \pi^T (D_\pi \hat{\xi})$$

In addition, it is related to ξ by $\xi(\pi) = \hat{\xi}(\pi, \mu_1, \mu_2, \ldots, \mu_M)$.

Proposition 5.2. ***[The SR(M-1) Condition]*** *Suppose that $\xi(\pi)$ is a collective demand function and $w(\pi)$ is the household's income function. Then, the extended Slutsky matrix is the sum of a symmetric matrix plus a matrix of rank at most $M-1$; that is,*

$$S(\pi) = \Sigma(\pi) + R(\pi)$$

where:

(1) The matrix $\Sigma(\pi)$ is symmetric and satisfies $v'\Sigma(\pi)v = 0$ for all vectors $v \in Span\{\pi\}$ and $v'\Sigma(\pi)v < 0$ for all vectors $v \notin Span\{\pi\}$.

(2) The matrix $R(\pi)$ is of rank at most $M-1$.

Proof. Since $\xi(\pi) = \hat{\xi}(\pi, \mu_1(\pi), \mu_2(\pi), \ldots, \mu_M(\pi))$. Then,

$$D_\pi \xi = D_\pi \hat{\xi} + \sum_{m=1}^{M-1} (D_{\mu_m} \hat{\xi})(D_\pi \mu_m). \tag{5.3}$$

Thus, the extended Slutsky matrix correspending to the collective demand function ξ is:

$$S(\pi) = D_\pi \xi - \frac{1}{\pi^T (D_\pi \xi)\pi} (D_\pi \xi)\pi\pi^T (D_\pi \xi)$$

Using equation (5.3), we get

$$S(\pi) = D_\pi \hat{\xi} + \sum_{m=1}^{M-1} (D_{\mu_m} \hat{\xi})(D_\pi \mu_m) - \frac{1}{\pi^T (D_\pi \xi)\pi} \left(D_\pi \hat{\xi} + \sum_{m=1}^{M-1} (D_{\mu_m} \hat{\xi})(D_\pi \mu_m) \right) \pi\pi^T (D_\pi \xi).$$

Rewrite this equation as:

$$\begin{aligned} S(\pi) &= D_\pi \hat{\xi} - \frac{1}{\pi^T (D_\pi \xi)\pi} (D_\pi \hat{\xi})\pi\pi^T (D_\pi \xi) \\ &\quad + \sum_{m=1}^{M-1} (D_{\mu_m} \hat{\xi})(D_\pi \mu_m) \left(I - \frac{1}{\pi^T (D_\pi \xi)\pi} \pi\pi^T (D_\pi \xi) \right) \\ &= \Sigma(\pi) + \sum_{m=1}^{M-1} a_m(\pi) b_m(\pi). \end{aligned}$$

Where the matrix $\Sigma(\pi)$ is the extended Slutsky matrix associated with the function $\hat{\xi}(\bullet, \mu_1, \ldots, \mu_M)$, the matrix $\Sigma(\pi)$ has the standard Slutsky properties, and where $a_m(\pi)$ and $b_m(\pi)$ are vectors defined by:

$$a_m(\pi) = D_{\mu_m}\hat{\xi}, \qquad b_m(\pi) = (D_\pi \mu_m)\left(I - \frac{1}{\pi^T(D_\pi \xi)\pi}\pi\pi^T(D_\pi \xi)\right).$$

In particular, $a_m(\pi)b_m(\pi)$ is of rank at most 1 for all $m = 1, \ldots, M-1$, so that $R(\pi) = \sum_{m=1}^{M-1} a_m(\pi)b_m(\pi)$ is of rank at most $M-1$. It follows that the extended Slutsky matrix $S(\pi)$ decomposes as the sum of a Symmetric matrix plus a matrix of rank at most $M-1$. □

Assume that the household's income function $w(\pi)$ is a non-homogeneous function, then the collective demand function $\xi(\pi)$ is also non-homogeneous.

Define the differential 1-form ω as follow:

$$\omega = \sum_{i=1}^{N} \xi^i d\pi_i - dw \tag{5.4}$$

its exterior derivative is

$$d\omega = \sum_{i,j=1}^{N} \frac{\partial \xi^i}{\partial \pi_j} d\pi_j \wedge d\pi_i$$

Introduce the vector field Π as

$$\Pi = \sum_{i=1}^{N} \pi_i \frac{\partial}{\partial \pi_i}.$$

Differentiating the budget constraint $\pi^T\xi = w(\pi)$, we get

$$\xi^i - \frac{\partial w}{\partial \pi_i} = -\sum_k \frac{\partial \xi^k}{\partial \pi_i}\pi_k = -\pi^T D_{\pi_i}\xi.$$

Then, the differential 1-form ω can be written as

$$\omega = \sum_{i=1}^{N} \xi^i d\pi_i - dw = -\sum_{i,k=1}^{N} \frac{\partial \xi^k}{\partial \pi_i}\pi_k d\pi_i.$$

Proposition 5.3. *Let $\xi(\pi)$ be a collective demand function of class C^2. Let ω be the differential 1-form defined above and $d\omega$ be its exterior derivative. Then, there exist $2M-1$ linearly independent 1-forms $\rho, \alpha_1, \ldots, \alpha_{M-1}, \beta_1, \ldots, \beta_{M-1}$ such that*

$$d\omega = \rho \wedge \omega + \sum_{m=1}^{M-1} \alpha_m \wedge \beta_m$$

Proof. Since

$$S(\pi) = \Sigma(\pi) + \sum_{m=1}^{M-1} a_m(\pi) b_m(\pi)$$

Then,

$$\begin{aligned} d\omega &= \sum_{i,j=1}^{N} \left(\frac{\partial x^i}{\partial \pi_j} - \frac{\partial x^j}{\partial \pi_i} \right) d\pi_i \wedge d\pi_j \\ &= \sum_{i,j=1}^{N} \left\{ \sum_{m=1}^{M-1} (a_m^i b_m^j - a_m^j b_m^i) + \left(v^i \left[\sum_k \frac{\partial \xi^k}{\partial \pi_j} \pi_k \right] \right. \right. \\ &\quad \left. \left. - v^j \left[\sum_k \frac{\partial \xi^k}{\partial \pi_i} \pi_k \right] \right) \right\} d\pi_i \wedge d\pi_j \end{aligned}$$

where vector v is defined as:

$$v = \frac{1}{\pi^T (D_\pi \xi) \pi} (D_\pi \xi) \pi.$$

So,

$$d\omega = \sum_{m=1}^{M-1} \alpha_m \wedge \beta_m + \rho \wedge \omega$$

where

$$\alpha_m = \sum_{i=1}^{N} a_m^i d\pi_i, \qquad \beta_m = \sum_{i=1}^{N} b_m^i d\pi_i$$

The proof is complete. □

Lemma 5.4. *[11] The following conditions are equivalent:*

(1) The Slutsky matrix $S(\pi)$ *decomposes as* $S(\pi) = \Sigma(\pi) + \sum_{m=1}^{M-1} a_m(\pi) b_m(\pi)$*, with* $\Sigma(\pi)$ *symmetric.*

(2) $\omega \wedge (d\omega)^M = 0$.

(3) There exists $2M-1$ *linearliy independent* 1*-forms* $(\rho, \alpha_1, \ldots, \alpha_{M-1}, \beta_1, \ldots, \beta_{M-1})$ *such that* $d\omega = \rho \wedge \omega + \sum_{m=1}^{M-1} \alpha_m \wedge \beta_m$.

The following theorem is a consequence of the Convex Darboux Theorem and solves the economic integration problem.

Theorem 5.5. *Let* $\xi(\pi)$ *be a* C^∞ *function that satisfies* $\pi'\xi(\pi) = w(\pi)$ *and the SR(M-1) condition in a neighbourhood,* $\mathcal{U}$*, of* $\bar{\pi}$*. Suppose that the matrix* $\Sigma(\pi)$ *is symmetric and negative definite on* $E(\pi)^\perp = span\{\xi - D_\pi w, D_{\mu_1}\hat{\xi}, \ldots, D_{\mu_{M-1}}\hat{\xi}\}^\perp$*. Then, there exist* M *positive functions* λ_m *and* M *strongly concave functions* V^m *such that*

$$\xi(\pi) = \sum_{m=1}^{M} \lambda_m D_\pi V^m(\pi) + D_\pi w$$

in some neighbourhood, $\mathcal{V}$*, of* $\bar{\pi}$*. Moreover, the function* $w(\pi)$ *is convex.*

Proof. Let ω_i and $\Omega_{i,j}$ be defined by

$$\omega_i = \xi^i - \frac{\partial w}{\partial \pi_i} \quad \text{and} \quad \Omega_{i,j} = \frac{\partial \xi^i}{\partial \pi_j} - \frac{\partial^2 w}{\partial \pi_j \pi_i}.$$

Then, $\Omega = D_\pi \xi - D_\pi^2 w$ can be written as

$$\begin{aligned} \Omega &= S(\pi) + \frac{1}{\pi^T (D_\pi \xi)\pi} (D_\pi \xi)\pi\pi^T (D_\pi \xi) - D_\pi^2 w \\ &= \Sigma(\pi) + \sum_{m=1}^{M-1} a_m(\pi) b_m(\pi) + \frac{1}{\pi^T (D_\pi \xi)\pi} (D_\pi \xi)\pi\pi^T (D_\pi \xi) - D_\pi^2 w. \end{aligned}$$

Introduce a subspace $E(\pi)$ as

$$E(\pi) = \{\xi - D_\pi w, D_{\mu_1}\hat{\xi}, \ldots, D_{\mu_{M-1}}\hat{\xi}\}.$$

The restriction of Ω to $E(\pi)^{\perp}$ is symmetric and negative definite. Since the matrix $\Sigma(\pi)$ is symmetric and negative definite on $E(\pi)^{\perp}$ and $D_{\pi}^{2}w$ is positive semi-definite on $E(\pi)^{\perp}$. The result follows from Convex Darboux Theorem. □

Theorem 5.6. *[11] A necessary and sufficient condition for a smooth differential 1-form $\omega = \sum_{i=1}^{N} \xi^{i} d\pi_{i} - dw(\pi)$ defined in a neighborhood; $\mathcal{U}$, of $\bar{\pi}$ to decompose into the sum $\omega = \sum_{i=1}^{M} f^{i} dg_{i}$, in a neighborhood; $\mathcal{V} \subset \mathcal{U}$, of $\bar{\pi}$, for some positive functions f^{i} and strongly concave functions g_{i}, is that there exist $2M-1$ linearly independent 1-forms $\alpha_{1}, \ldots, \alpha_{M-1}, \beta_{1}, \ldots, \beta_{M-1}, \gamma$ such that $d\omega$ decompose as*

$$d\omega = \omega \wedge \gamma + \sum_{i=1}^{M-1} \alpha_{i} \wedge \beta_{i}$$

in a neighborhood;$\mathcal{V} \subset \mathcal{U}$, of $\bar{\pi}$, and the matrix

$$\Omega(\bar{\pi}) = D_{\pi}\xi(\bar{\pi}) - D_{\pi}^{2}w(\bar{\pi})$$

is symmetric and negative definite on $[E(\bar{\pi})]^{\perp}$, where

$$E(\pi) = Span\{\omega, \beta_{1}, \ldots, \beta_{M-1}\}.$$

5.2 Collective Demand Function: Homogeneous Case.

In this section, we consider problem $\mathcal{F}$ with the additional assumption that the collective income function $w(\pi)$ is 1-homogeneous and the Pareto weights $\mu_{m}(\pi), \forall m = 1, \ldots, M$ are 0-homogeneous; that is,

$$D_{\pi}w(\pi)\pi = w(\pi), \quad \text{and} \quad D_{\pi}\mu_{m}(\pi)\pi = 0, \quad \forall m = 1, \ldots, M.$$

As before, the problem $\mathcal{F}$ can be written as a two stage maximization problem

$$\max_x U(x,\mu) \quad \text{subject to} \quad \pi' x \leq w(\pi) \tag{P1}$$

where

$$U(x,\mu) = \max_{y_1,y_2,\ldots,y_M,Y} \left\{ \sum_{m=1}^{M} \mu_m(\pi) U^m(y_1,y_2,\ldots,y_M,Y) | y = x \right\} \tag{P2}$$

The first-order conditions for $(P1)$ are

$$U_x = \lambda(\pi)\pi$$

$$\pi^T x \leq w(\pi)$$

where $\lambda(\pi) > 0$ is the Lagrange multiplier associated with the constraint. Define the function $\hat{V}(\pi,\mu)$ as

$$\hat{V}(\pi,\mu) = \max_\xi \{U(\xi,\mu) | \pi^T \xi \leq w(\pi)\}.$$

So the collective indirect utility function is defined as

$$V(\pi) = \hat{V}(\pi,\mu(\pi)) = U(\xi(\pi),\mu(\pi))$$

Using envelope theorem, we find that

$$\frac{\partial V}{\partial \pi_i} = \lambda(\pi)\left(\frac{\partial w}{\partial \pi_i} - \xi^i\right) + \sum_{m=1}^{M} \frac{\partial U}{\partial \mu_m}\frac{\partial \mu_m}{\partial \pi_i}$$

The homogeneity assumption on $w(\pi)$ implies that the collective demand function $\xi(\pi)$ is 0-homogeneous and the collective indirect utility function $V(\pi)$ is 0-homogeneous and the Lagrange multiplier function $\lambda(\pi)$ is -1-homogeneous; that is,

$$D_\pi \xi(\pi)\pi = 0, \quad D_\pi V(\pi)\pi = 0, \quad \text{and} \quad D_\pi \lambda(\pi)\pi = -\lambda(\pi).$$

In this setting, the Slutsky matrix takes the form

$$S(\pi) = D_\pi \hat{\xi} - \frac{1}{\pi^T (D_\pi \xi) \pi} (D_\pi \hat{\xi}) \pi \pi^T (D_\pi \xi) + \sum_{m=1}^{M-1} (D_{\mu_m} \hat{\xi})(D_\pi \mu_m).$$

We notice that the Slutsky matrix of the collective demand function $\xi(\pi)$ decomposes as the sum of a symmetric matrix plus a matrix of rank at most $M-1$. Call this condition SRH(M-1).

Define the 0-homogeneous differential 1-form ω as

$$\omega(\pi) = \sum_{i=1}^{N} (\xi^i - \frac{\partial w}{\partial \pi_i}) d\pi_i.$$

and the vector field Π as

$$\Pi = \sum_{i=1}^{N} \pi_i \frac{\partial}{\partial \pi_i}$$

Proposition 5.7. *Let $\xi(\pi)$ be a collective demand function, $V(\pi)$ be the collective indirect utility function and let $V^m(\pi)$ be the indirect utility function for member m. Then,*

$$\frac{\partial V}{\partial \pi_i} = \lambda(\pi) \left(\frac{\partial w}{\partial \pi_i} - \xi^i \right) + \sum_{m=1}^{M-1} \left(V^m(\pi) - V^M(\pi) \right) \frac{\partial \mu_m}{\partial \pi_i}.$$

Proof. We know that the collective demand function $\xi(\pi)$ is the solution of the following maximization problem

$$\max_{\xi} U(\xi, \mu(\pi)) \quad \text{subject to} \quad \pi^T \xi \leq w(\pi).$$

The envelope theorem implies that the derivative of the function $V(\pi)$ with respect to π_i is given by

$$\frac{\partial V}{\partial \pi_i} = \lambda(\pi) \left(\frac{\partial w}{\partial \pi_i} - \xi^i \right) + \sum_{m=1}^{M} \frac{\partial U}{\partial \mu_m} \frac{\partial \mu_m}{\partial \pi_i}$$

But

$$U(\xi, \mu(\pi)) = \sum_{m=1}^{M} \mu_m(\pi) U^m(y_1, \ldots, y_M, Y).$$

Using the normalization condition $\sum_{m=1}^{M} \mu_m(\pi) = 1$, we can write $U(\xi, \mu(\pi))$ as

$$U(\xi, \mu(\pi)) = \sum_{m=1}^{M-1} \mu_m(\pi)(U^m(y_1, \ldots, y_M, Y) - U^M(y_1, \ldots, y_M, Y)) + U^M(y_1, \ldots, y_M, Y).$$

It follows that

$$\frac{\partial U}{\partial \mu_m} = U^m(y_1, \ldots, y_M, Y) - U^M(y_1, \ldots, y_M, Y) = V^m(\pi) - V^M(\pi).$$

Thus,

$$\frac{\partial V}{\partial \pi_i} = \lambda(\pi)\left(\frac{\partial w}{\partial \pi_i} - \xi^i\right) + \sum_{m=1}^{M-1} \left(V^m(\pi) - V^M(\pi)\right) \frac{\partial \mu_m}{\partial \pi_i}.$$

□

Then we can decompose ω as

$$\omega = \frac{-1}{\lambda(\pi)} dV(\pi) + \sum_{m=1}^{M-1} \phi^m d\mu_m.$$

where $\phi^m = \frac{V^m(\pi) - V^M(\pi)}{\lambda(\pi)}$. Notice that ω is decomposed as

$$\omega(\pi) = \sum_{m=1}^{M} a^m(\pi) du_m(\pi)$$

where the functions $a^m(\pi)$ are 1-homogeneous and the functions $u_m(\pi)$ are 0-homogeneous.

5.3 Individual Excess Demands

In many applications, it can be helpful to consider the excess demand functions instead of Marshallian demand functions. So, for any Marshallian demand function $x(p)$, there exists an excess demand function $z(p)$ defined by $z(p) = x(p) - e$ where $e \in \mathbb{R}^n_+$ is the initial endowment. In [10], the authors give the local and global characterization of excess demand functions. In [4,6], Aloqeili generalizes the characterization conditions of excess demand functions of Geanakoplos and polemarchakis. In this section, we solve the homogeneous mathematical integration problem and economic integration problem of excess demand function $z(p)$.

Let $U(x)$ be a consumer utility function over a set of consumption bundles that satisfy certain smoothness, monotonicity, and concavity conditions and let the Marshallian demand function $x \in \mathbb{R}^n_+$ sloves the individual maximization problem

$$(\mathcal{Z}) \left\{ \max_x U(x) \quad \text{subject to} \quad p^T x = p^T e. \right.$$

where e is the initial endowment and $p \in \mathbb{R}^n_{++}$ is the price vector that is associated with the consumption bundle x. Note that the income function $w(p) = p^T e$ is homogeneous of degree one. The excess demand function is defined by $z(p) = x(p) - e$ where $x(p)$ solves the problem $\mathcal{Z}$. Note that $z_p = x_p$. Then, the Jacobian matrix $z_p(p)$ satisfies

$$z_p = \lambda U^{-1} - \frac{\lambda}{p^T U^{-1} p}(U^{-1}p)(U^{-1}p)^T - \frac{U^{-1}p}{p^T U^{-1} p} z^T(p) \tag{5.5}$$

where U is the Hessian matrix of $U(x)$. Thus, the problem $\mathcal{Z}$ can be written as follow:

$$\max_z u(z) \quad \text{subject to} \quad p^T z = 0$$

where $u(z) = U(z+e)$. The Slutsky matrix $\hat{S} = x_p - \frac{x(p)}{w(p)}(p^T x_p)$ takes the form:

$$z_p + \frac{1}{p^T e}(z(p)+e)(z^T(p))$$

Since $z(p)z^T(p)$ is a symmetric matrix, the Slutsky matrix of excess demand functions is a symmetric matrix $\hat{s}$ defined as follow:

$$\hat{s} = z_p + \frac{1}{p^T e} e z^T(p)$$

Proposition 5.8. *Let $z(p)$ be an excess demand function. The matrix $\hat{s}$ has the following properties:*

(1) $\hat{s} = \hat{s}^T$.

(2) $p^T\hat{s} = \hat{s}p = 0$.

(3) The matrix $\hat{s}$ is negative semi-definite.

(4) The matrix $\hat{s}$ has rank $n-1$.

The first-order conditions for maximum are

$$u_z = \lambda(p)p$$

$$p^T z(p) = 0$$

where $\lambda(p) > 0$ is the Lagrange multiplier associated with the constraint. Introduce the indirect utility function $V(p)$ defined by

$$V(p) = \max_z \{u(z) | z \in \mathbb{R}^n, p^T z \leq 0\}$$

By the envelope theorem, we get

$$V_p(p) = -\lambda(p)z(p) \tag{5.6}$$

Proposition 5.9. *Let $V(p)$ be an indirect utility function, $u(z)$ is reguler. Then, $V(p)$ is quasi-convex, positively homogeneous of degree zero.*

Define 0-homogeneous differential 1-form ω as follow:

$$\omega = \sum_{i=1}^{n} z^i(p)dp_i \tag{5.7}$$

Introduce the vector field π as

$$\pi = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}$$

Then, equation (5.6) implies that the differential 1-form ω that is defined in (5.7) can be represented in the form

$$\omega = \mu dV$$

where the function μ is negative and homogeneous of degree one and the function V is quasiconvex and homogeneous of degree zero.

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