Approximate Solutions for Bending of Beams and Buckling of Columns Made of Functionally Graded Materials

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Approximate Solutions for Bending of Beams and Buckling of Columns Made of Functionally Graded Materials

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ABSTRACT

In this research approximate solutions are developed to solve Functionally Graded Material (FGM) beam and column problems. FGM is considered in the transversal direction of both beams and columns using different functions; power function, exponential function, and sigmoidal function. Also, different power values are used to consider different material concentrations in the transversal direction of beams and columns. Approximate solutions are first developed for FGM beams using iterative techniques based on averaging the elasticity modulus in beam's tension and compression zones to determine different structural outputs like deflection, slope, shear and moment diagrams. Then outputs are compared to classical Euler bending theory to validate the adopted approximate method used in the solution. In addition to beam bending problems, approximate solutions for FGM column buckling problems are also developed. The methods adopted in solution are Rayleigh's Quotient, Timoshenko's Quotient, and Rayleigh-Ritz method. These methods are compared to the analytical classical Euler buckling solution. MATLAB program is adopted to solve both problems since it is simple and precise. The main results of this study shed the light on the importance of approximate solutions in solving FGM bending and buckling problems. For bending problems, the approximate method suggested resulted in trivial error (~0%) when compared to analytical solution. As for buckling problems, Rayleigh-Ritz method showed to be the most accurate one in calculating critical buckling load with error less than 0.75%, while Rayleigh method led to significant errors (~22%). At the end of this work, some recommendations were presented for future research work on FG structural members.

Keywords: Functionally Graded Materials, Bending of Beams, Buckling of Columns, Approximate solution.

Chapter One: Introduction

1.1 Background

The development of new materials is considered an engineering concern overtime. The characteristics of the new engineering products are very important for their use in construction purposes and for the costs of the new material produced (Karamanli, 2016). The stability of the new product is considered one of high concern in engineering (Jarachi *et al*., 2019). The problems associated with the new products depend on the original components, the structure and the shape of the new product called functionally graded material (FGM). The first time the FGM introduced was in 1984 in Japan in aerospace project (Niino *et al*., 1987). This project required the provision of material with thickness 10 mm and tolerate outside temperature 2,000 K and inside temperature 1,000 K. The efforts continued on the development of FGM. The activities of FGM developing included civil engineering, mining, chemical, and biomedical products (Zhang *et al*., 2019). FGM provide the opportunity to produce smooth materials with different chemical, mechanical and physical properties such as Poisson's ratio, Young's Modulus, coefficient of thermal expansion, density and Shear Modulus. Moreover, the graduality and its variability provided new properties of FGM including stiffness ratio, corrosion resistivity, thermal conductivity, hardness, and specific heat (Allahyarzdeh *et al.*, 2016).

The structure of beams and columns affect its stability. In civil engineering the structural material of the beam or the column will affect the bending of beams and bucking of columns (Alshafei, 2013). In addition, the type of loading has a great effect on the structural response of structural members (Temsah *et al*., 2018[a], Temsah *et al*., 2018[b], Jahami *et al*., 2019, Jahami *et al*., 2020, Jahami *et al*., 2021). Structures were classified into two types: isotropic and orthotropic. Isotropic structures have homogeneous mechanical and thermal properties in all directions. In this case measuring the bending of beams and buckling of columns will be direct. However, Orthotropic structures have different mechanical and thermal properties in each direction. According to the previous methods and specifications, different criteria of beams and columns require definition to calculate bending and buckling. For beams, the definition of cross section can be used to calculate the moment of inertia, also the neutral axis and distances from extreme fibers to the neutral axis are required too. For columns, buckling load factors require to be estimated. The results of the buckling load factor will depend on the structure of the column and the type of functionally graded materials used (Jarachi *et al*., 2019). To reach proper solution derivations required to solve the different situations of the material structure in different directions. The objective of this research is to find solutions for bending and buckling of functionally graded columns using different methods and analysis procedures.

1.2 Problem Statement

The wide variety of civil engineering products of FGM increased the problems associated with the calculations of FGM product specifications. One of the problems is associated with stability of FGM products (Matsunaga, 2009). The stability of FGM structural members is related to several outputs like buckling stress, natural frequencies and distribution of displacements (Sofiyey, 2009). This requires to determine the mechanical and physical characteristics of any new FGM member and to find solutions for the problems associated with tolerability and forces distribution in complex structures (Swaminathan and Naveenkumar, 2014).

Usually, the classical methods to solve FGM structural elements stability problems may be complicated since they were derived for isotropic materials with assumptions and simplifications. Bending of beams has been investigated using different analytical methods like Euler-Bernoulli method, Timoshenko method (Karamanli, 2016; Huang and Li, 2010), and Reddy-Bickford method (Karamanli, 2016). For buckling problem, different analytical methods were used to find solutions for FGM columns. The most familiar buckling result was calculated using Timoshenko beam theory (TBT). Huang and Li (2016) used TBT to solve the buckling of functionally graded circular columns and the shear deformation. Also, Kiani and Eslami (2012) used the TBT to find the buckling resulted from exposure to different temperatures. Many other researchers followed these methods to solve FGM column problems like Jarachi (2019) Storch and Elishakoff (2018), Alshabatat (2018), and Karamanli, (2016).

In order to simplify the analysis of FGM elements (beams and columns), approximate solutions must be developed. This can be done by following many famous approximate methods in structural analysis. For beam bending problems, the approximate solution can be developed by averaging the elasticity modulus above and below neutral axis. This can be done by trying several iterations with preliminary assumptions. As for column buckling problems, approximate solutions can be developed following three main methods; Rayleigh’s Quotient, Timoshenko’s Quotient, and Rayleigh-Ritz method. These methods can be efficient in solving FGM problems and reduce calculation time and complexity.

1.3 Research Significance

Although many studies have discussed the analytical solutions for FGM beams and columns, most of these solutions were complex and hard to be determined for structural elements with irregular shapes and material distribution. Therefore, this work aims to provide approximate solutions for bending and buckling problems for FGM beams and columns. These solutions will lead to accurately determine different beam outputs like displacement, slope, and shear and moment diagrams at different locations. As for columns, it will lead to determine different column outputs like lateral displacement and critical buckling loads. The provided solutions will be direct and simple compared to the classical analytical solutions which will allow engineers that work with FGM beams and columns to analyze and later design these structural members easily.

1.4 Research Aim and Objectives

The major aim of conducting this work is to derive approximate solutions to solve FGM beam and column problems. To achieve this aim, the analytical solutions of both bending and buckling problems will be first introduced. Then, several approximate solutions will be derived and validated by comparing its outputs with the analytical solution ones. Finally, a set of recommendations for practicing engineers working on FGM using approximate solutions will be provided as well as future research ideas in this field.

1.5 Research Questions

Once the work of this thesis is conducted, it is expected that the following questions are answered:

I. Can approximate solutions be simple and accurate when analyzing bending of FG beams?

II. Can approximate solutions be simple and accurate when analyzing buckling of FG columns?

III. Are approximate solutions for bending and buckling problems valid for different boundary conditions and material properties?

1.6 Research Hypothesis

This work is achived to prove the validity and effectviness of some hypotheses. On the other hand, it will prove the nullity and annulment of other hypotheses. The hypotheses assumed in this work are:

A) Null Hypothesis, H_o – which assumes that approximate solutions are not effective for bending and buckling problems and states: "approximate solutions are neither simple nor accurate for analyzing both bending and buckling problems"

B) Hypothesis One, H_1 – which assumes the opposite of null hypothesis, and states: "approximate solutions are simple and accurate for analyzing both bending and buckling problems"

C) Hypothesis Two, H_2– which assumes that approximate solutions for bending and buckling problems are valid for different boundary conditions and material properties

D) Hypothesis Three, H_3 – which assumes the opposite of Hypothesis two, and states: "approximate solutions for bending and buckling problems are not valid for different boundary conditions and material properties"

E) Hypothesis Four, H_4 – which assumes that all Euler – Bernoulli assumptions apply to FG beams and columns. These assumptions include the following:

1- Beam cross section is always perpendicular to the bending line.
2- Shear deformation in beams is neglected
3- Euler – Bernoulli beam model appears stiffer than the actual model
4- Euler – Bernoulli method is only applicable to slender beams (long and thin beams)

1.7 Thesis Statement

This research is entitled: "Approximate Solutions for Bending of Beams and Buckling of Columns Made of Functionally Graded Materials". In this thesis, bending and buckling problems will be solved using simple approximate methods which will be compared to the classical analytical methods to check its validity and efficiency.

1.8 Thesis Structure

This thesis consists of five chapters that will cover a series of topics regarding the FGMs and their key contributions in protecting beams and columns from any opportunity of failure. The structure of the thesis consists of:

- **Chapter one – INTRODUCTION**

In chapter 1, an illustration regarding the work background, research aim and objectives, research structure, thesis questions, problem statement, and research hypothesis will be addressed. In addition, chapter 1 will offer further information on work limitations.

- **Chapter two – LITERATURE REVIEW**

Chapter 2 will discuss the main analytical and approximate methods adopted to solve FG beam and column problems in previous research works.

- **Chapter three – RESEARCH METHODOLOGY**

Chapter 3 will discuss the methodology of this work followed to accomplish the aim and objectives of the thesis. In addition, chapter 3 will address major variables and constraints used and assumed to accomplish this work.

- **Chapter four – RESULTS AND DISCUSSIONS**

In this chapter, the major findings of this work will be represented. Furthermore, chapter 4 will address the discussions, and relate the findings of this work to other articles.

- **Chapter five – CONCLUSIONS AND RECOMMENDATIONS**

In this chapter, major conclusions and recommendations of this work will be represented. Moreover, chapter 5 will provide a number of suggestions related to future work regarding the FGMs and their key characteristics for determining opportunities and locating areas for scientists, scholars, and engineers, in which further improvements o beams and columns performance can be attained.

1.9 Thesis Limitations

It is worth to mention that in this work the need to meet different professors and academic staff in Israa University to consult them on a number of information and inquiries required to be answered regarding the FGMs was hard, due to COVID-19 regulations of social distancing and working at home for most of academic employees. Therefore, instead of face-to-face meeting in Israa University, all inquiries were answered via e-mails, and WhatsApp applications, whereas meeting with professors was accomplished via online platforms, such as Zoom and Google Meet.

Chapter Two: Literature Review

2.1 Aim and Introduction

This chapter demonstrates the major work done on isotropic and functionally graded structural elements. Both beams and columns will be considered in this literature. Also, analytical solutions and approximate solution adopted by researchers are included. The most recent studies were considered to make sure that all modern methods of analysis were properly covered. This will help in determining the gap in the literature to consider it in the current study.

2.2 Literature Review

Karamanhi (2016) investigated the electrostatic deformation of functionally graded beams applied on different boundary conditions. The study used the beam method and symmetric smoothed particle hydrodynamics (SSPH). Euler-Bernoulli, Reddy-Bickford and Timoshenko beam theories were used for calculations. The results showed that SSPH provided logic results convergence rate for the problem. The author recommends that SSPH can be used to solve bending problems for FGM beams

The static behavior of functionally graded metal–ceramic (FGM) beams at ambient temperature is studied by Kadoli *et al.* (2007) using a displacement field based on higher order shear deformation theory. FGM beams with varying metal or ceramic volume fractions based on power law exponents are explored. The finite element form of the static equilibrium equation for FGM beam is provided using the notion of stationary potential energy. Thus, two stiffness matrices are produced, one of which reflects the impact of normal rotation and the other shear rotation. The numerical findings for transverse displacement, axial and shear stresses in a fairly thick FGM beam under uniform distributed load for clamped–clamped and simply supported boundary conditions are thoroughly examined. The influence of the power law exponent on the deflection and stresses of various metal–ceramic FGM beam combinations is also discussed. The investigations show that the static deflection and static stresses in the beam do not remain constant depending on whether the loading is on the ceramic rich face or the metal rich face of the beam.

Ashirbekov *et al.* (2020) conducted a study to investigate the main characteristics of a beam that is made out of FGM. The beam was analyzed through finite element modeling (ANSYS) and compared to classical analytical solutions. The method suggested by the authors can be applied for complex 2D and 3D geometries having different distributions of functionally graded materials. Results indicate high accuracy between analytical and finite element solution of 2D and 3D geometries.

Zamorska (2014) modified Euler-Bernoulli theorem, to work for beams having varying cross-section of $A(x)$, and varying moment of inertia, $I(x)$. The equation was written as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] + \rho A(x)\frac{\partial^2 u}{\partial t^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}1)$$

Where “u” is the beam deflection, “ρ” is the beam density and “E” is the modulus of elasticity.

This equation is applied to a beam that has several initial conditions, assuming that the end of the beam is fixed at the origin, as illustrated in Figure (2-1). Taking into account that the function of beam deflection, u, can be represented through the function: $u(x,t) = y(x)e^{i\omega t}$, then, the formula of motion can be rearranged to be:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] - \rho\,\omega^2 A(x) y = 0 \tag{2-2}$$

Where, ω is the natural frequency. Assuming that the cross-sectional area and the moment of inertia are polynomials, as:

$$A(x) = A_o\left(\frac{\alpha-1}{L}x+1\right)^2, I(x) = I_o\left(\frac{\alpha-1}{L}x+1\right)^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{2-3}$$

Where $A_o = A(0)$, and $I_o = I(0)$, and α does not equal to 1, which is the proportional coefficient related to the cross-section of the beam, and $x \in [0, L]$, then for $a(x) = \left(\frac{\alpha-1}{L}x+1\right)^{-1}$, a_1,a_2, a_3, and a_4 can be expressed as:

$$a_1(x) = -\Omega^4 a^2(x), a_2(x) = 0, a_3(x) = \frac{12(\alpha-1)}{L}\, a^2(x), a_4(x) = \frac{8(\alpha-1)}{L} a(x) \tag{2-4}$$

Where Ω denotes the beam's vibration frequency, which can be represented through the formula:

$$\Omega^4 = \frac{\omega^2 \rho A_o}{EI_o} \tag{2-5}$$

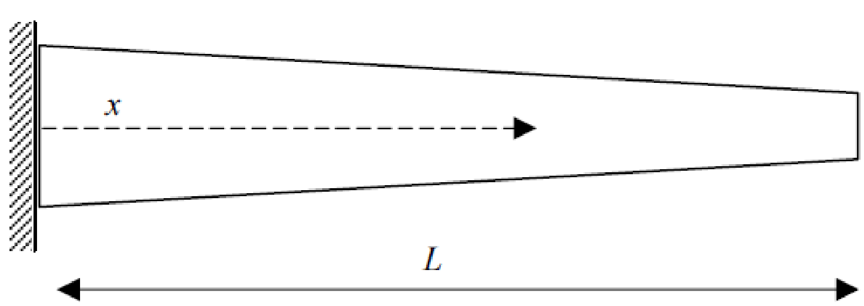


Figure 2- 1: dimensions of the beam considered by Zamorska (2014)

Hosseini-Hashemi et al. (2016) conducted a study to determine the mechanical properties of functionally graded beams via power and exponential laws. Different power and exponential coefficient were used. It was found that natural frequencies were highly affected by the variation of sections. Also, similar effect was realized for beam mode shapes. Power and exponential functions were written by the authors as follows:

$$X(z) = (X_1 - X_2)\left(\frac{h-2z}{2h}\right)^p + X_1 \rightarrow Power\ function \qquad \textbf{\textit{(2-6)}}$$

$$X(z) = X_b e^{\emptyset z} \rightarrow Exponential\ function \qquad \textbf{\textit{(2-7)}}$$

Where, "X_1" is the material property for the first material, "X_2" is the material property for the second material, "z" is the height of the beam under consideration, "h" is the beam depth and "∅" illustrates the material gradation corresponding to the thickness value represented in via exponential function coefficient.

Akbarzadeh and Shariati (2016) conducted a study to analyze the mechanical properties for sigmoidal functionally graded beams (S-FGBs). They use a number of relations related to sigmoidal principles in FGMs in beams including:

$$\mathrm{X}(z) = \left(1 - {}^{1}\!/_{2}\left(\frac{\left[\frac{h}{2}\right]-z}{h/2}\right)^p\right)X_1 + \left(\frac{1}{2}\left(\frac{\frac{h}{2}-z}{h/2}\right)^p\right)X_2, \qquad 0 \le z \le {}^{h}\!/_{2} \qquad (2\text{-}8)$$

And:

$$X(z) = \left({}^{1}\!/_{2}\left(\frac{\left[\frac{h}{2}\right]+z}{h/2}\right)^p\right)X_1 + \left(1 - \frac{1}{2}\left(\frac{\frac{h}{2}+z}{h/2}\right)^p\right)X_2, \qquad -\frac{h}{2} \le z \le 0 \qquad (2\text{-}9)$$

Ruhi *et al*. (2005) investigated the characteristics and key role of FGMs in resolving columns buckling. A thermo-elastic analysis was conducted on a thick wall cylinder, having a finite length, as illustrated in Figure (2-2). Stress, strain, and displacement graphs were plotted across the cylinder thickness, and through the length of the cylinder. It was shown that the graduation of cylinder FGMs components is considered to be a significant factor that affects the thermos-mechanical response of the cylinder made of FGMs.

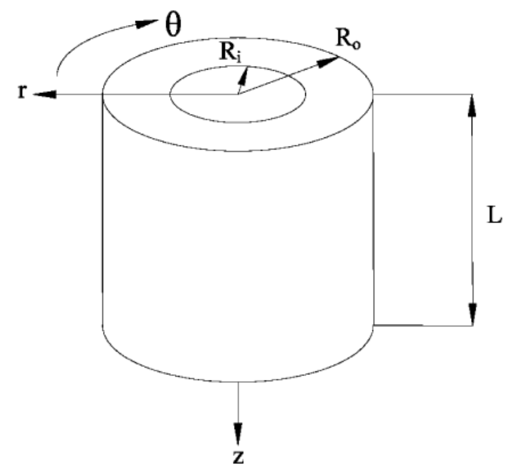


Figure 2- 2: dimensions of a cylinder having finite thick length.

Source: (Ruhi et al., 2005).

Jarachi *et al*. (2019) developed a model for axially loaded columns as per Euro code 2. The study aimed at enabling engineers to find solution of stability of compressed columns without using computer software. This helps in the evaluation of critical buckling load (P_{cr}) through the calculation of the influence of different parameters on stability of columns. The study applied the non-buckling analysis and the middle cross section methods. The load duration was calculated for columns considering the length of the column.

Bajc *et al.* (2015) conducted a study to determine the buckling of reinforced concrete column using semi-analytical procedures under the exposure of fire. The concentration was on the buckling time and load. The results showed that the load-carrying capacity reduce as the time to fire exposure increased. Moreover, the load had small effect on buckling load capacity.

Weng *et al.* (2020) introduced a solution of buckling resistance for reinforced concrete columns during inelastic deformation. The equilibrium differential equation was used to the inelastic buckling loads of columns. The method introduced was found to be rational and can be used effectively to measure buckling resistance of columns when compared to eccentrically loaded columns.

Aydogdu (2008) studied the vibration and buckling of axially functionally graded supported beams through the application of semi-inverse method. The analysis was depending on Euler-Bernoulli theory. The results showed that young's modulus changes exponentially between the edges of the beam for the vibration and the buckling problem.

Huang and Li (2010) studied the circular cylindrical column stability for functionally graded materials considering shear deformation. The study applied new approach away of the assumption of uniform shear stress at cross-section applied using Timoshenko beam theory. They applied approach depending on traction-free surface condition. They derived equations for deflection and rotation. Comparison between the two approached was carried out. The new approach approved results that are comparable with the Timoshenko beam theory and Reddy-Bickford beam theory.

Ramkumar and Ganesan (2008) calculated buckling and vibration of box columns under different thermal conditions. The box columns are thin walled and the isotropic and functionally square plates were used. Finite-element method using classical plate theory was applied to reach the results. The buckling analysis were carried out at different boundary conditions under different material types. The composition of material (isotropic or graded plates) affects the buckling temperature. The buckling temperature increased with non-isotropic box columns (mix of ceramic and metal). They concluded that the constituents of columns affect its buckling in different temperature conditions.

Singh *et al.* (2009) studied the buckling analysis of both uniform and non-uniform functionally graded columns in axial direction. They used low-dimensional mathematical model. The results showed that the model used was able to calculate the buckling load of columns for both uniform and non-uniform columns. The model enables to calculate the area modulus of elasticity and rigidity.

Yilmaz *et al.* (2013) conducted research on analyzing buckling for axially functionally graded non-uniform columns using localized differential quadrature method. The method used was able to find weighting coefficients for differential quadrature using neighboring points in forward and backward types. The boundary conditions were implemented into weighting coefficients matrices. The results of this method were compared to other methods and it was found to be of high potential to reach solutions for non-uniform functionally graded columns.

Parmar & Thakkar (2017) conducted a study to investigate the properties and the process used in making FGMs. The method used in their study was reviewing a wide range of literature review to determine the key properties of FGMs and their role in reducing the failure in beams and in columns. They found that FGMS can be used in a wide range of applications, such as automobiles – in combustion chambers, engine cylinder liners, diesel engine pistons, and flywheels. FGMs can be also used for submarine, commercial and industrial, and aerospace applications. In addition, they found that FGMs can be made through three major steps, which are: weighing and mixing the powders of two or more different materials, ramming and stacking these powders, and sintering the overall material.

Alshabatat (2018) conducted a study to evaluate methods, through which buckling capacity of slender columns can be improved, and to assess the key role of FGMs in solving buckling problem of slender columns and determine whether if they are more effective than isotropic materials. For accomplishing the goal of his study, Alshabatat conducts a finite element analysis using genetic algorithm and uses Mori-Tanaka method for finding the optimal solution. His findings indicate the significance of FGMs in supporting slender columns and minimize the amount of buckling failure. In addition, he found that his results can be effectively used for designing columns with very low value of buckling via integrating FGMs.

Darılmaz *et al.* (2015) investigated a method, through which the technical gap on the elastic buckling behaviors of columns through the use of FGMs grid system can be filled. In addition, the goal of the study is to cover the gap of few literature articles that discuss FGMs with in-plane loads. To achieve their study goal, Darılmaz et al. investigate a hybrid stress modelling of columns via finite element analysis, and solutions through the use of FGMs to reduce the buckling issue were proposed. The results of the numerical analysis indicate that aspect ratios and material gradations of FGMs play a vital role in minimizing the amount of buckling behavior related to columns. In addition, they believe that their finite element results would be highly effective for designers to utilize FGMs for resolving buckling issue in columns.

Maleki & Mohammadi (2017) investigated the impact of cracks on the strength of columns that are made of FGMs, and to investigate the opportunity of these columns to have buckling and failure when they are subjected to piezo-electric patches. For achieving their study goal, Maleki & Mohammadi modeled the cracks through a rotational spring that has negligible mass. Then, they applied a number of constraints and boundary conditions on the locations of piezo-electric patches and locations, where cracks occur. The results of their study revealed that cracks in a column, which uses FGMs could highly minimize its capability to carry loads and can cause buckling depending on the size of the crack, depth, and its location. In addition, it was found that piezo-electric patches generate local torques, which in case are controlled and monitored, the amount of crack and its depth can be reduced.

Ranganathan *et al.* (2015) investigated the failure related to buckling in slender columns that use FGMs, and determine whether if FGMs can reduce the amount of buckling in these columns. For achieving their study goal, they examine numerous types of columns having advanced materials such FGMs and microstructure materials, assuming a number of constraints and boundary conditions. In addition, they determine the amount of buckling load through Rayleigh-Ritz approach, finite element analysis, and linear perturbation analysis. The results indicate that the use of FGMs could change the opportunity of buckling occurrence in comparison to conventional homogenous columns, which has corresponding flexural modulus. In addition, it was found that in case FGMs are not correctly chosen this can result in a 66% minimization of the capacity of carrying loads. On the other hand, a correct selection of FGMs can promote the capacity of carrying loads and reduce the opportunity of buckling by twenty-two percent.

Many studied were conducted on the approximate solutions of FG beams subjected to statics load conditions. Niknam *et al.* (2014) investigated FG beams having tapered sections subjected to mechanical and thermal loading conditions. The governing equations were constructed, and the possibility of achieving an analytical solution is discussed. In the absence of axial force along the beam, a closed form solution to the issue is offered. The Galerkin approach is used to overcome the analytical solution's inadequacy in the general situation with axial force. Furthermore, the Generalized Differential Quadrature (GDQ) approach is used to discretize and solve the governing equations in their general form, as well as to validate the findings produced by the other two methods. Varied analytical and computational methodologies are used to study the impact of various thermal and mechanical stress on the nonlinear bending of a tapered FG beams.

The stress and deformation behavior of a shear deformable functionally graded cantilever beam utilizing the B-spline collocation technique was reported by Mahapatra *et al.* (2019). The material grading was along the length of the beam and changes according to the power law. The Poisson's ratio was taken to be a constant. To establish a unified formulation for Timoshenko beams, the equations were obtained utilizing the virtual work approach in the framework of Timoshenko beams. For approximation, a sixth order basis function is utilized, and collocation points are created using Greville abscissa. Deformation and strains, as well as bending (axial) and transverse (shear) stresses and the location of the neutral axis, are investigated for a wide variety of power law index values. The findings are presented along the cross-section and length of the beam.

Xu *et al.* (2014) studied displacement and stress distribution of tapered simply supported FG beams. The exponential law was adopted for the graded elasticity modulus of the beam. The generic formulations for the displacements and stresses of the beam under static loads, which perfectly satisfy the controlling differential equations and boundary conditions, are analytically calculated out on the basis of two-dimensional elasticity theory. The Fourier sinusoidal series expansions to the boundary conditions on the upper and lower surfaces of the beams are used to estimate the unknown coefficients in the solutions. The effect of altering Young's modulus rules on the displacements and stresses of functionally graded beams is thoroughly examined. The derived two-dimensional elasticity solution may be used to evaluate the applicability of different approximate solutions and numerical approaches for the previously specified functionally graded beams.

As for buckling of FG columns, many researchers conducted studied on the approximate solutions to determine the critical buckling loads. Ranganathan *et al.* (2015) determined the critical buckling load of columns using multiple functional distributions for the flexural modulus. A limitation was placed to guarantee that the volume averaged flexural modulus for all columns was similar in order to have a meaningful comparison. To address the eigenvalue issue, the linear perturbation approach and the Rayleigh-Ritz method were utilized, and the findings produced were compared to those found in the literature. When compared to a homogeneous column with same flexural modulus, functional gradation was shown to either boost or reduce a column's buckling load bearing capability. In example, it was proven that a bad choice of functional gradation might result in a 66% loss in load bearing capacity, whilst a good choice could increase it by 22%. Finally, by ensuring that the normalized spatial distribution of the flexural modulus agrees with the normalized mode shape, the critical buckling load of any column may be maximized.

Huang and Li (2011) proposed a unique analytic technique for resolving the buckling instability of Euler-Bernoulli columns with arbitrary axial non-homogeneity and/or variable cross section. The governing differential equation for buckling of columns with variable flexural stiffness is reduced to a Fredholm integral equation for various columns such as pinned-pinned columns, clamped columns, and cantilevered columns. By demanding that the resultant integral equation have a nontrivial solution, the critical buckling load may be precisely calculated. The method's usefulness is demonstrated by comparing their findings to previous closed-form solutions and numerical data. Flexural rigidity can be represented by a wide range of functions, including polynomials, trigonometric and exponential functions, among others. Examples are presented to demonstrate how to improve the load-carrying capacity of tapered columns for acceptable form profiles with constant volume or weight, and the proposed approach is useful for optimal column design against buckling in engineering applications. This approach may be developed to deal with the free vibration of nonuniform beams with axially varying material characteristics.

Wang and Pilkey (1986) developed approximate reanalysis methods based on the generalized Rayleigh quotient. The reanalysis equations developed in this research are straightforward algebraic equations. These relationships can be utilized to quickly generate an estimated value of the modified system for reanalysis. If more precise answers are required, the proposed method can be supplemented with previous nonlinear eigenvalue problem reanalysis formulations. The current solution only works for the basic mode. It should be attempted to extend this strategy to higher modes. Furthermore, because only the lowest eigenvalues are frequently of importance in buckling issues, the current technique is ideal for buckling load reanalysis.

The traditional energy approach for calculating the estimated critical buckling loads of bars was addressed by Ioakimidis (2018). This approach is based on the bar's stability condition and the suitable selection of an approximation to the bar's deflection. Furthermore, when determining the critical buckling load, it is commonly connected to the Rayleigh quotient or the Timoshenko quotient. The energy method was utilized to determine the critical buckling loads of bars but with new computational methodology. "Mathematica" was used to solve buckling problems using the modern computational method of quantifier elimination which minimizes both Timoshenko's quotient and Rayleigh's quotient. This method, which eliminates partial differentiations, is also more rigorous than the conventional method based on partial derivatives since it does not need the employment of the requirements for a minimum based on second partial derivatives, which are commonly neglected in reality. Furthermore, it is quite simple to utilize within Mathematica's rich computing environment. The current technique was demonstrated in a number of bar buckling issues, including parametric buckling problems. Buckling issues of bars with two internal unilateral constraints are also investigated, where the classical energy approach is difficult to use. Even in this demanding application, the essential buckling load is estimated immediately and with acceptable precision.

2.3 Chapter Summary

Through reviewing different articles, it was found that the potential of FGMs is huge, as they contribute to many advantages in several engineering applications, which include automobiles, aerospace, commercial purposes, industrial uses, and mechanical applications.

Analytical solutions for bending and buckling problems were extensively investigated and covered in the literature. Also, many approximate solutions were developed to analyze bending and buckling problems when having complex geometry or irrigular loading conditions. This is so important as a start point in our study on approximate solutions for FG beams and columns and will help us to develop simple and easy solutios for bending and buckling problems.

Chapter Three: Research Methodology

3.1 Aim and Introduction

This chapter aims to derive approximate solutions for the analysis of FG beams and columns in order to compare the structural behavior of both types and provide the necessary recommendations. Several factors will be taken into consideration while deriving the equations such as: the change in beams' and columns' dimensions, the change in the type of materials used, the change in boundary conditions. The MATLAB program will be adopted to solve all equations in order to compare the findings of this study with analytical methods to validate the work.

3.2 Methodology of Work

This work will go through several stages before reaching the final conclusion. First, a literature review study will be conducted on analytical and approximate solutions for bending and buckling problems in order to deeply understand these topics before starting this research work. The most recent references will be relied upon as they are the latest in any scientific field and carry the summary of all previous studies. Therefore, the focus will be mainly on the research completed in the last ten years.

The next step will be to derive the equations for approximate solutions for both bending and buckling problems. All requirements necessary for the formation of these equations will be defined, such as the nature of the loads and their values, material properties, and boundary conditions. The constants and variables will be determined for each equation to be defined later in MATLAB program, which will help to know the expected inputs and outputs from MATLAB in each equation.

Then, MATLAB models will be validated to make sure they represent each case considered in this study. The comparison will be done with respect to analytical methods and several outputs will be compared like: displacements, slopes, stresses, and strains. This will be done for both bending and buckling problems. After validation process, further analysis will be conducted to evaluate each type of structural elements which will lead to some recommendations for future research work.

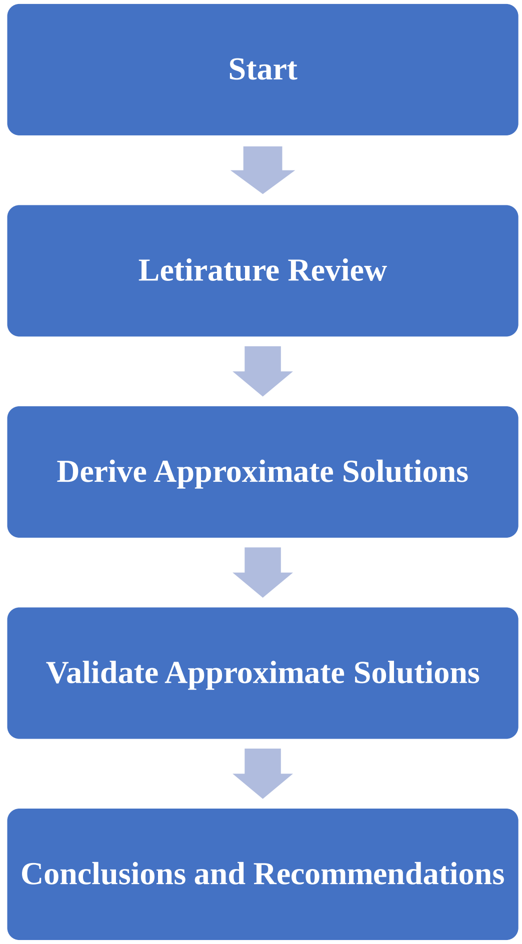


Figure 3- 1: Research Methodology followed in this thesis.

3.3 Analytical Solutions for Functionally Graded Material Members

3.3.1 Beams Subjected to Bending

In this section, the major equations for functionally graded beams will be derived to be used in the MATLAB later. Consider a beam made from two materials as shown in Figure 3-2. The beam is simply supported and subjected to a distributed load of intensity "q_0" (Figure 3-3). From static concept, the summation of axial forces acting on the beam must be zero. Hence, the following equation can be written:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma(z).dA = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varepsilon(z).E(z).b.dz = 0 \tag{3-1}$$

Where "$\sigma(z)$" is the stress at depth z, "dA" is the area of a small element in the beam as shown in Figure (3-2), "E(x)" is the elasticity modulus at depth z, "ε(x)" is the strain at depth z, and "b" is the beam width.

Another equation can be formed is related to the summation of internal moments in the section which must be equal to the external applied moment. In this case the moment is considered at the midspan and equals to $\frac{q_0 L^2}{8}$. Therefore, the following equation can be written:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} z.\varepsilon(z).E(z).b.dz = \frac{q_0.L^2}{8} \tag{3-2}$$

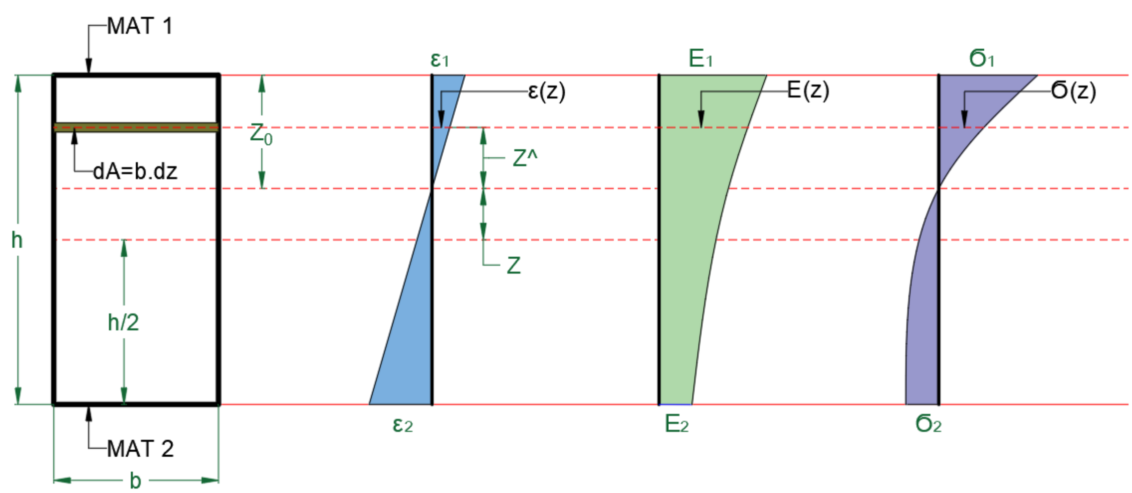


Figure 3- 2: Stress and strain distribution for functionally graded beam.

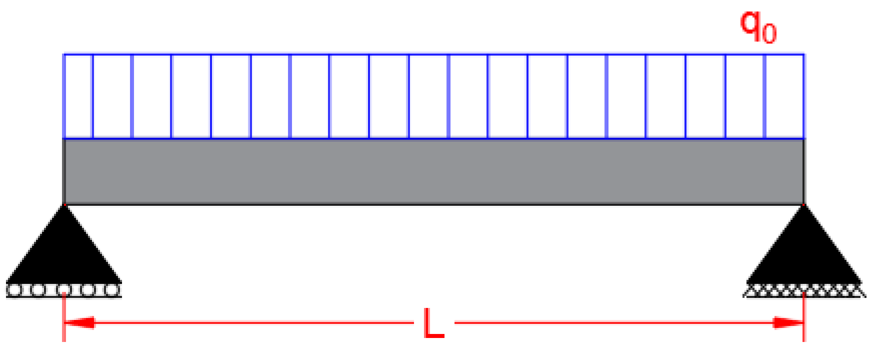


Figure 3- 3: Functionally graded beam under distributed loading.

According to Figure 3-2, the compatibility of strain can be used as follows:

$$\frac{\varepsilon(z)}{\hat{z}} = \frac{\varepsilon_1}{z_0} \rightarrow \varepsilon(z) = \frac{\hat{z}.\varepsilon_1}{z_0}$$

(3-3)

Where "ε_1" is the maximum strain at the upper metal surface, "z_0" is the neutral axis depth measures from top, and "$\hat{z}$". Also, "$\hat{z}$" can be written as $z + z_0 - \frac{h}{2}$, substituting this in equation 3-3 leads to the following relation:

$$\varepsilon(z) = \frac{\varepsilon_1}{z_0}\left(z + z_0 - \frac{h}{2}\right) \qquad (3\text{-}4)$$

Another relation can be used which is the power law. This law relates the elasticity modulus at depth z "E(z)" to the elasticity modulus of both materials (upper and bottom surface materials) and reflects the uniformity of material distribution through the depth. The power law can be written as follows:

$$E(z) = E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)^k \qquad (3\text{-}5)$$

Where "E_1" is the elasticity modulus for the upper surface material, and "E_2" is the elasticity modulus for the lower surface material. By substituting equations 3-4 and 3-5 in equations 3-1 and 3-2, and for k=1, they become:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \left[E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)\right]\left[\frac{\varepsilon_1}{z_0}\left(z + z_0 - \frac{h}{2}\right)\right] b.dz = 0 \tag{3-6}$$

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} z.\left[E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)\right]\left[\frac{\varepsilon_m}{z_0}\left(z + z_0 - \frac{h}{2}\right)\right] b.dz = \frac{q_0.L^2}{8} \tag{3-7}$$

Solving equations 3-6 and 3-7 lead to the magnitude of z_0 and ε_1. Therefore, the stress at level z can be determined as follows:

$$\sigma(z) = E(z).\varepsilon(z) \tag{3-8}$$

As for deflection equations for FGB, they will be considered at the mid depth of the beam (z=0). Therefore, the strain at this level will be:

$$\varepsilon_{xx} = \frac{d\theta}{dx} = -z\frac{d^2y}{dx^2} \tag{3-9}$$

According to Hook's law:

$$\sigma_{xx} = E(z).\varepsilon_{xx} = -\left[E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)^k\right].z\frac{d^2y}{dx^2} \tag{3-10}$$

Substituting equation 3-5 in equation 3-7 leads to:

$$M_{xx} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)^k\right].z^2\frac{d^2y}{dx^2}dz = -D_{xx}\frac{d^2y}{dx^2} \tag{3-11}$$

Where D_{xx} is:

$$D_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[E_2 + (E_1 - E_2)\left(\frac{z}{h} + \frac{1}{2}\right)^k\right] . z^2\, dz \tag{3-12}$$

This will lead to the following deferential equation:

$$\frac{d^2}{dx^2}\left(D_{xx}\frac{d^2y}{dx^2}\right) = w(x) \tag{3-13}$$

Where w(x) is the load function, "σ_{xx}" is the normal stress along the longitudinal axis of the beam (at z=0), and "ε_{xx}" is the normal strain along the longitudinal axis of the beam (at z=0).

Other methods can be used to determine the elasticity modulus of functionally graded material beams like exponential and sigmoidal method. According to exponential method E(z) will be as follows:

$$E(z) = E_2 e^{-\partial\left(1-\frac{2z}{h}\right)} \tag{3-14}$$

As for the sigmoidal method, E(z) will be as follows:

$$E(z) = \begin{matrix} \left(1 - {}^1\!/_2\left(\frac{\left[\frac{h}{2}\right]-z}{h/2}\right)^p\right)E_2 + \left(\frac{1}{2}\left(\frac{\frac{h}{2}-z}{h/2}\right)^p\right)E_1, & 0 \le z \le {}^h\!/_2 \\ \left({}^1\!/_2\left(\frac{\left[\frac{h}{2}\right]+z}{h/2}\right)^p\right)E_2 + \left(1 - \frac{1}{2}\left(\frac{\frac{h}{2}+z}{h/2}\right)^p\right)E_1, & -\frac{h}{2} \le z \le 0 \end{matrix} \tag{3-15}$$

Where "∂" is the exponential gradation, "p" is the power coefficient.

3.3.2 Columns Subjected to Buckling

The solution for buckling problems in the case of functionally graded columns is similar in the form to isotropic columns. For the case of simply supported column (Figure

3-4), the lateral displacement and buckling load will be according to Taeprasartsit (2011) as follows:

$$y(x) = d_z\left(\cos\left(\sqrt{\frac{-P}{D_{xx}}}x\right) + \frac{1-\cos\left(L\sqrt{-P/D_{xx}}\right)}{\sin\left(L\sqrt{-P/D_{xx}}\right)}\sin\left(\sqrt{\frac{-P}{D_{xx}}}x\right) - 1\right) \quad (3\text{-}16)$$

$$P = \frac{\pi^2 D_{xx}}{L^2} \quad (3\text{-}17)$$

As for the case of fix supported column (Figure 3-5), the buckling load according to Taeprasartsit (2011) will be as follows:

$$P = \frac{4\pi^2 D_{xx}}{L^2} \quad (3\text{-}18)$$

Where:

$$d_z = \frac{(E_1 - E_2)hk}{2(k+2)(E_1 + kE_2)} \quad (3\text{-}19)$$

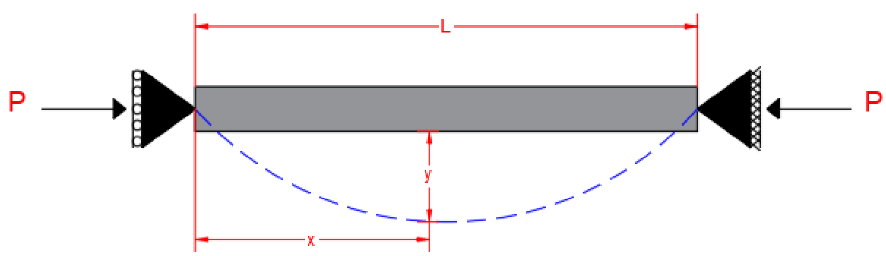


Figure 3- 4: The simply supported column considered in the case study. (Taeprasartsit, 2011)

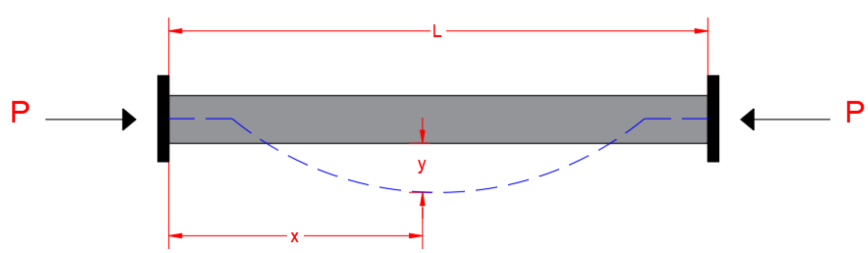


Figure 3- 5: The fix supported column considered in the case study. (Taeprasartsit, 2011)

3.4 Approximate solutions for bending and buckling problems

3.4.1 Beams Subjected to Bending

The derivation of approximate solution of beam bending is described in Figure 3-14. This method is based on using average value for elasticity modulus of the upper surface and bottom surface materials. First of all, Using the compatibility of strains, the following equation can be written:

$$\epsilon_2 = \frac{h - Z_0}{Z_0} \epsilon_1 \tag{3-20}$$

By taking the equilibrium of forces (C and T), the following equation can be reached:

$$(h - Z_0)^2 = {Z_0}^2 \frac{f_1 E_1}{f_2 E_2} \tag{3-21}$$

By taking the equilibrium of moment at the centerline, the following relation can be determined between Z_0 and ε_1 as follows:

$$\epsilon_1 = \frac{3 q_0 L^2}{f_1 E_1 b h Z_0} \tag{3-22}$$

The terms f_1E_1 and f_2E_2 can be determined as follows:

$$f_1 E_1 = \frac{\int_{Z_0}^{h/2} E(z) dz}{Z_0} \tag{3-23}$$

$$f_2 E_2 = \frac{\int_{-\frac{h}{2}}^{Z_0} E(z) dz}{h - Z_0} \tag{3-24}$$

E(z) can be determined by power, exponential, or sigmoidal methods as described earlier.

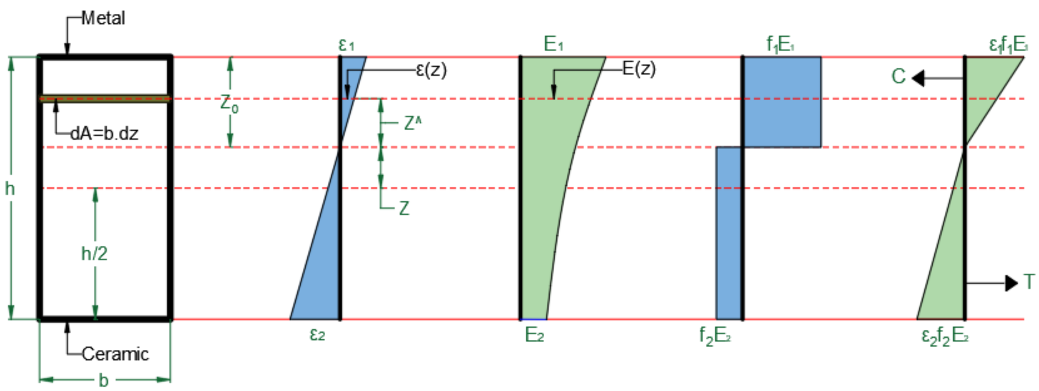
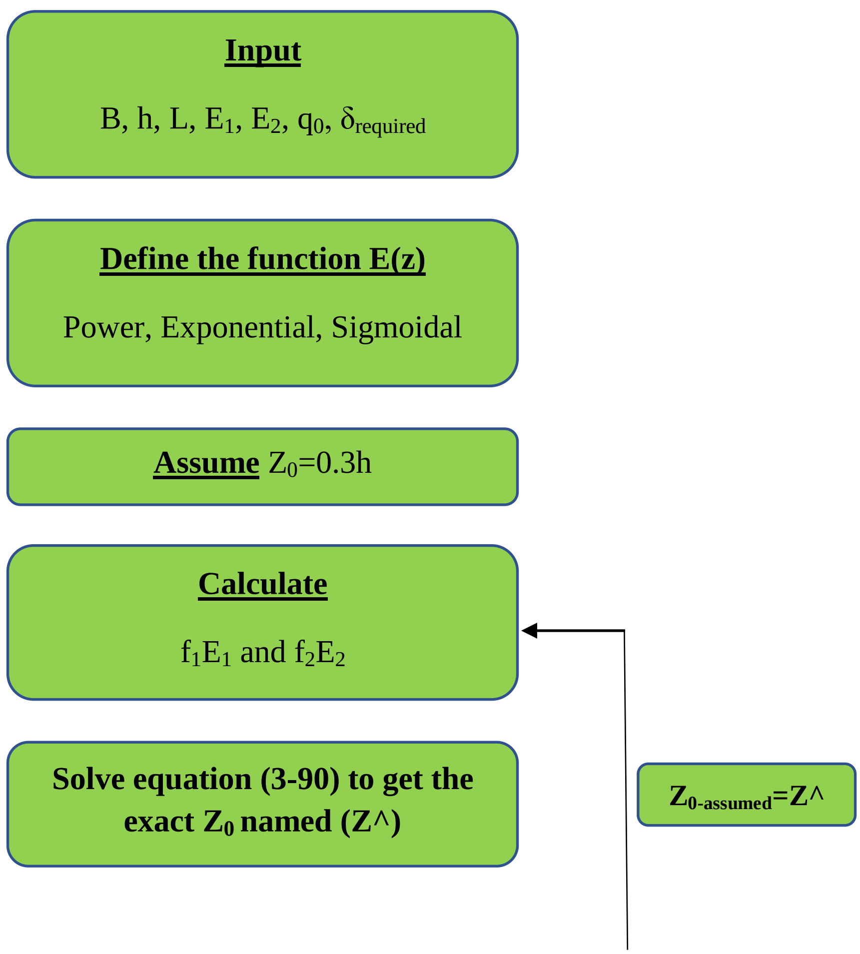


Figure 3- 6: The Derivation of approximation method for bending.

Using a specialized software like MATLAB, strain values can be determined as shown in Figure 3-7.



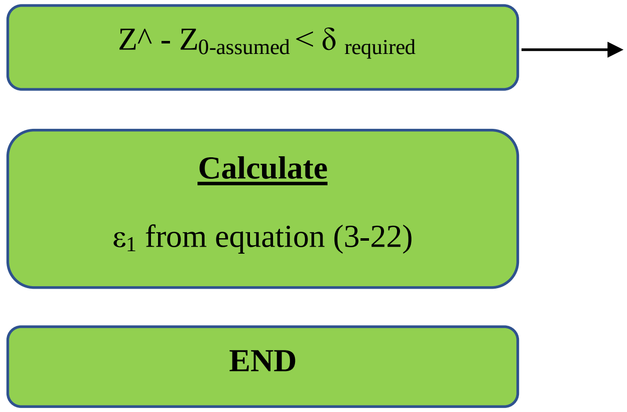


Figure 3-7: The proposed program in MATLAB to determine strain components using approximate approach.

3.4.2 Columns Subjected to Buckling

There are many approximate solutions for buckling problems. Three main methods will be described in this section: the Timoshenko approach, the Rayleigh approach, and the Rayleigh – Ritz approach.

First let's derive Timoshenko's approach equations (Westergaard, 1878). Consider the column shown in Figure 3-8. Assume "z" the deflection in the starting shape, "y" the additional deflection from the starting shape to the final shape, which means the total final deflection y + z, and "EI" is the flexural stiffness.

Assume that the bending moment has a value of $S = R.z$ at the starting shape. This will lead to $dS = dR.z$. Since an infinitesimal element with length dx is used in the derivation, the following relation can be written to determine the deflection:

$$-\frac{d^2y}{dx^2}dx = \left(\frac{R.z}{EI} + \frac{d^2z}{dx^2}\right)dx \qquad (3\text{-}25)$$

Since the deflection at support levels is zero, the leads to the following relation:

$$dU = \int_0^l \left(\frac{R.z}{EI} + \frac{d^2z}{dx^2}\right) dx.z.dR = 0 \tag{3-26}$$

Where "dU" is the complimentary energy increment for the infinitesimal element. This leads to the following relation for the force "R":

$$R = -\frac{\int_0^l z.\frac{d^2z}{dx^2}dx}{\int_0^l \frac{z^2 dx}{EI}} \tag{3-27}$$

By choosing the parabola as a starting shape:

$$z = \frac{4c}{l^2}(lx - x^2) \tag{3-28}$$

The R will have the following value:

$$R = \frac{10\ EI}{l^2} \tag{3-29}$$

Timoshenko improved equation 3-27 to take the following form:

$$P = \frac{\int_0^l \left(\frac{dz}{dx}\right)^2 dx}{\int_0^l \frac{z^2 dx}{EI}} \tag{3-30}$$

Where "P" is Timoshenko's variant of the energy method.

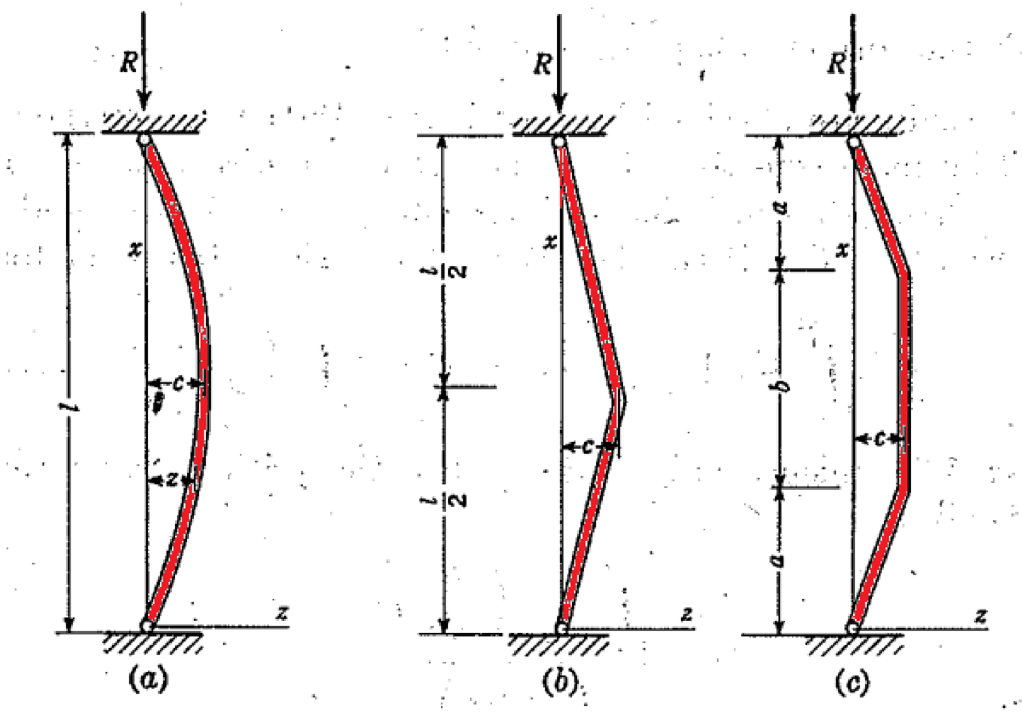


Figure 3-8: The parameters considered in deriving Timoshenko's quotient method. (Westergaard, 1878)

Now let's derive the main equations for Ryleigh method (Welleman, 2014). Consider the in Figure 3-9 for the derivation. At initial stages and as the load is increased gradually before the buckling phenomena takes place, the strain energy will be as follows:

$$E = \int_0^l \frac{1}{2} EA \in^2 dx \tag{3-31}$$

After buckling, bending effect will appear in the energy equation as follows:

$$E = \int_0^l \frac{1}{2} EA \in^2 dx + \int_0^l \frac{1}{2} EIk^2 dx \tag{3-32}$$

After buckling, the change in strain energy will result from the bending component only as shown:

$$\Delta E = \int_0^l \frac{1}{2} EIk^2 dx = \int_0^l \frac{1}{2} EI \left(\frac{d^2w}{dx^2}\right)^2 dx \tag{3-33}$$

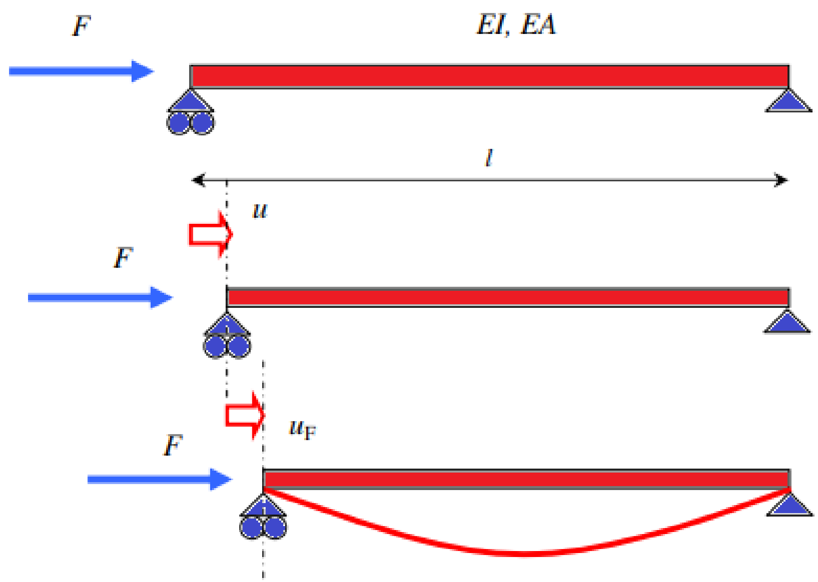


Figure 3- 8: The parameters considered in deriving Rayleigh's quotient method. (Welleman, 2014)

This strain energy is equal to the work performed by the concentrated load (P) when the column is transferred from the straight position to the bending position. This work is W=F x μ_f. This horizontal displacement can be expressed in the vertical deflection using Pythagoras theorem as shown in Figure (3-9)

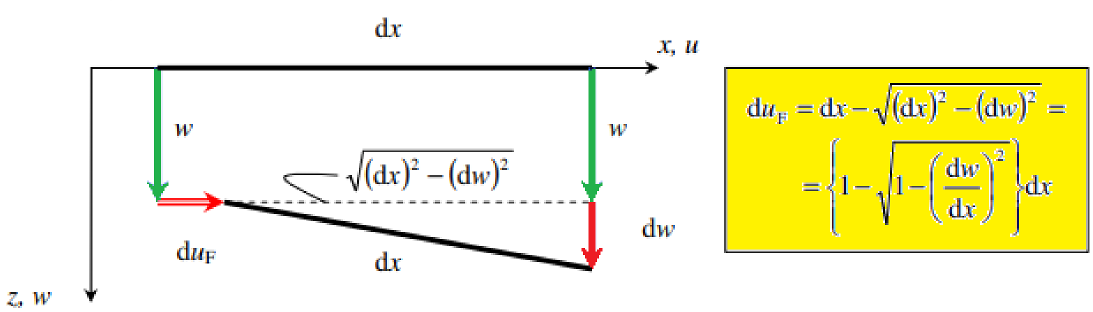


Figure 3- 9: The relation between horizontal and vertical deflection (Rayleigh's quotient). (Welleman, 2014)

Then the total horizontal displacement will be equal to:

$$\mu_f = \int_0^l \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \tag{3-34}$$

Let ΔE = A will result in the following equation:

$$\int_0^l \frac{1}{2} EI \left(\frac{d^2w}{dx^2}\right)^2 dx = F \times \int_0^l \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \tag{3-35}$$

Where "F" is the buckling load. By assuming a displacement field w(x) we can find a buckling load which is usually higher than the real buckling load. The better the assumed displacement field the closer the buckling load will be to the real value.

As for Rayleigh – Ritz approach, the energy functional F is defined as the difference between the maximum kinetic energy and the maximum potential energy (Kumar, 2017). The deflection function W is chosen in terms of trial functions Øi as follows:

$$W = \sum_{i=1}^{N} c_i \emptyset_i \tag{3-36}$$

Where "c_i" is an arbitrary coefficient and "$Ø_i$" is a function that can take any form (polynomial, exponential, trigonometric, ..etc.). The factor c_i can be determined by minimizing the energy functional F by taking partial derivatives with respect to c_i $\frac{\partial F}{\partial c_i} = 0$. As for the trial function it can be expressed as follows:

$$\emptyset_i = f_i \prod_{J=1}^{Ne} B_J{}^{SP}, \;\; i = 1 \rightarrow N \tag{3-37}$$

Where "f_i" is ith term of the complete set of polynomials, "Ne" is the edges number, "B_J" is the boundary expression of the jth edge, and "SP" is the suitable power.

For a beam, the maximum strain energy will be equal to:

$$E_s = \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx \qquad (3\text{-}38)$$

The maximum kinetic energy will be equal to:

$$K_e = \frac{\rho A\omega^2}{2}\int_0^l w^2 dx \qquad (3\text{-}39)$$

Where "ρ" is the density, "A" is the area, "ω" is the circular frequency. Assuming X=x/l and W=w/l, and equating maximum strain and kinetic energies we obtain:

$$\int_0^l \left(\frac{d^2W}{dX^2}\right)^2 dX - \Omega^2 \int_0^l W^2 dX = 0 \qquad (3\text{-}40)$$

Where:

$$\Omega = \omega L^2 \sqrt{\frac{\rho A}{EI}} \qquad (3\text{-}41)$$

This will lead to the following eigen value problem:

$$[[K]_{N\times N} - \Omega^2 [M]_{N\times N}][C]_{Nx1} = 0 \qquad (3\text{-}42)$$

Where:

$$k_{ij} = \int_0^l \frac{d^2\emptyset_i}{dX^2} \times \frac{d^2\emptyset_j}{dX^2} dX \qquad (3\text{-}43)$$

$$m_{ij} = \int_0^l \emptyset_i \emptyset_j dX \qquad (3\text{-}44)$$

3.5 Software Package

MATLAB will be adopted to model each study case in this thesis. Modeling inputs for every case can be provided through coding like: span length, cross sectional dimensions, applied loads, material gradation, ...etc. Then all required mathematical operations for both analytical and approximate solutions can be entered to MATLAB using special functions like: integration functions, derivation functions, and differential equation functions. This will lead to the outputs to be extracted for each case. Different results can be gathered like: displacement, slope, bending moment, shear force, critical buckling load, … etc. These results will come in different forms like tables and diagrams. The whole procedure on MATLAB will save much time and effort which will lead to consider different study cases to validate approximate solutions for bending and buckling problems in FG beams and columns.

3.6 Study Variables

There are a number of variables and parameters that are taken into account for modelling and analysis of FG columns and beams. These parameters and variables defined in MATLAB include:

a) Dimensions of beams and columns (Span, length, and width). Several span depth ratios will be considered in this study (L/h = 10, 20, and 40).
b) Load pattern and type (static uniformly distributed load).
c) Boundary condition (Simply supported and Fixed supported).
d) Different material combinations. Three material combinations will be considered in this study; Aluminum – Silicon, SuS304 – ZrO_2, and Ti-6Al-4V – Si_3N_4

By defining these variables, MATLAB will analyze each problem using both analytical and approximate methods defined. Then, outputs can be compared to validate approximate solution methods and prove their efficiency in solving FGM problems easily and accurately.

3.7 Chapter Summary

This chapter discussed analytical and approximate method equations for bending and buckling problems in details. First the methodology of work was defined, then the equations of analytical solutions for bending and buckling problems were mentioned. After that, approximate solution equations were derived for both bending and buckling problems. Then, analysis procedure was described and MATLAB abilities and input/output requirements were also mentioned. Finally, the study variables were mentioned in details to be discussed later in chapter 4.

Chapter Four: Results and Discussion

4.1 Aim and Introduction

This chapter will discuss the analysis results done on both functionally graded beams and columns using both analytical and approximate methods. For beams and columns, two supporting conditions were considered for both beams and columns: simply supported and cantilever. As for the section dimensions, three span – depth ratios were considered: 10, 20, and 40. In addition, the effect of different material functions was considered too (power, exponential, and sigmoidal) along with different power values: 0, 1, and 10. The findings from both analytical and approximate methods will be compared to prove the efficiency of approximate solutions suggested by the author.

4.2 Functionally Graded Beams

4.2.1 Validation of MATLAB Codes

The codes built for the analytical research work done by Karamanli (2016) were validated in order to take them as a reference for later comparisons with approximate solutions. The maximum deflection at different span – depth ratios was determined by MATLAB codes and compared with two analytical solutions adopted by Karamanli (2016) which are Timoshenko and reddy-bickford. Karamanli considered two materials in his FG beam; Aluminum (Al) with elasticity modulus of 70 GPa and Silicon (Si) with elasticity modulus of 151 GPa. The comparison is summarized in Table 4-1. The results were closed to each other with error percentage less than 1% which means that the analytical solution modeled in MATLAB is able to describe the behavior of FGB successfully, and ready to be compared with approximate solutions later.

Table 4- 1: Maximum deflection results for the present study compared to Karamanli (2016)

Method of work	Maximum deflection at different span - depth ratio (L/h)		
	L/h=10	L/h=20	L/h=50
Timoshenko Beam Theory	0.1449	1.1383	17.6929
Reddy-Bickford Beam Theory	0.1413	1.1101	17.27
Present study	0.1414	1.1312	17.6755

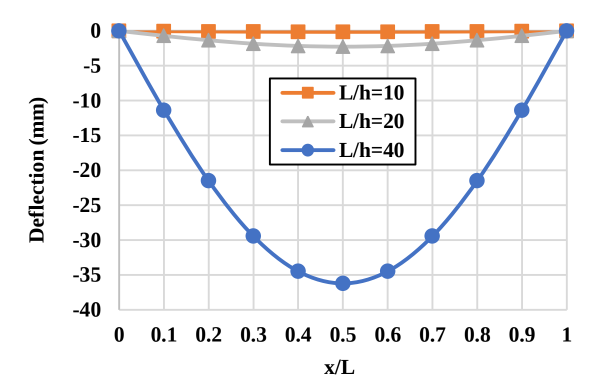
4.2.2 Effect of Span – Depth (L/h) Ratio

Three Span to depth (L/h) ratios were considered in this study: 10, 20 and 40.

Figure 4-1 depicts the deflection diagram at each distance ratio (x/L) measured for the simply supported beam. It can be seen that as span – depth ratio increases, the deflection of simply supported beam at any point increases. For example, the maximum deflection for the beam with (L/h=40) was 36.2 mm, while for (L/h=10) the maximum deflection was 0.14 mm.

Similar trend can be realized for slope results as shown in Figure 4-2. As the L/h ratio increases, the maximum slope at supports increases. For instance, the maximum slope at supports for the beam with L/h=40 was 0.03 rad while for the beam with L/h=20 it was 0.0036 rad. As we get closer to the beam mis span, the slope decreases until reaching zero at x/L=0.5 (the mid span).

As for the interaction forced (bending moment and shear force), results showed that as the L/h ratio increases, both bending moment and shear force increased. The beam with L/h=40 had a maximum moment of 0.021 N.m compared to 0.0014 for the beam with L/h=10 as depicted in Figure 4-3, and a maximum shear of 0.022 N compared to 0.0054 N for the beam with L/h=10 as depicted in Figure 4-4.



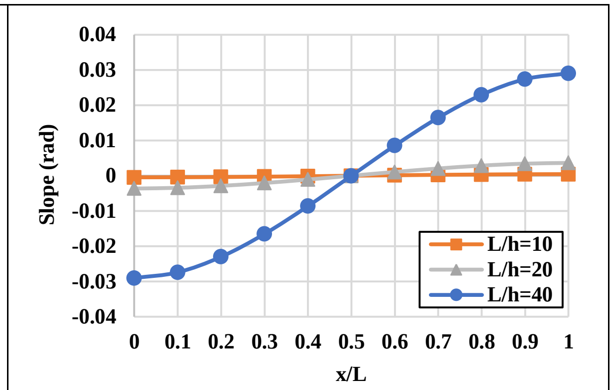
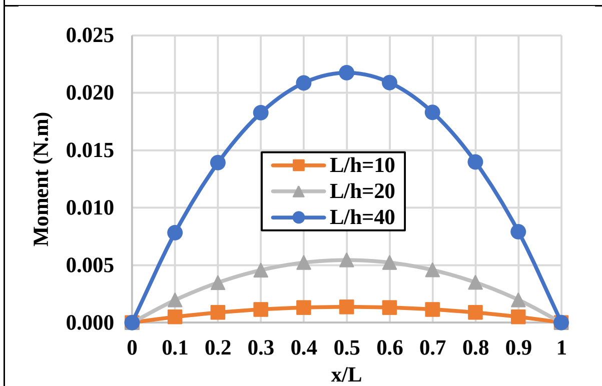


Figure 4- 1: Deflection for simply supported beam with different L/h ratios.

Figure 4- 2: Slope for simply supported beam with different L/h ratios.



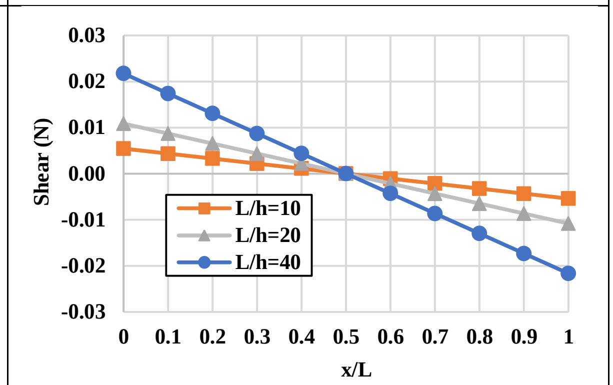
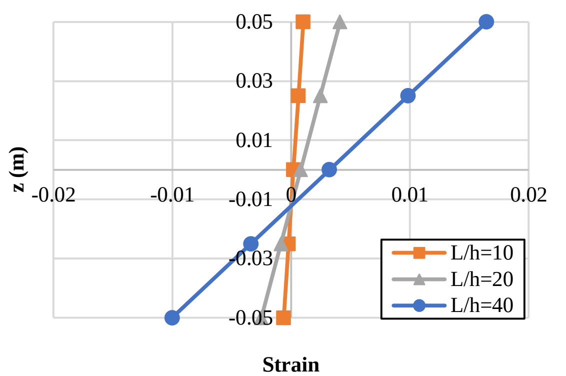


Figure 4- 3: Bending moment for simply supported beam with different L/h ratios.

Figure 4- 4: Shear for simply supported beam with different L/h ratios.

Figures 4-5 and 4-6 show the strain and stress distribution for simply supported beams with different span – depth ratio. It can be seen that as the span – depth ratio increased, the maximum stress values at top and bottom surfaces increased. This is due to higher moment effects for larger spans which led to higher normal stress magnitudes.



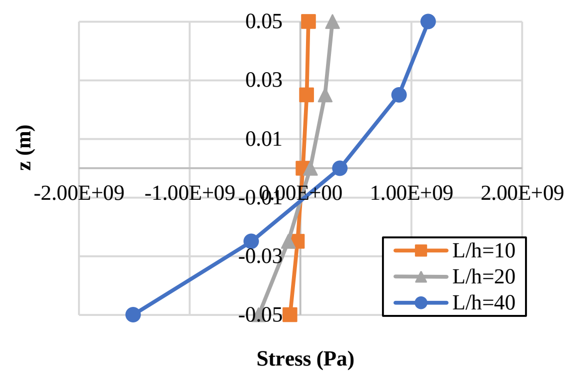


Figure 4- 5: Strain distribution for simply supported beam with different L/h ratios.

Figure 4- 6: stress distribution for simply supported beam with different L/h ratios.

The same effect was studied for cantilevered beams. Although the behavior of cantilevered beams is different due to different boundary conditions, the same trend was realized for different L/h ratio. According to Figure 4-7, as the L/h ratio increases, the maximum deflection at the tip increases. For example, the maximum deflection for the beam with (L/h=40) was 347 mm compared to 1.36 mm for the beam with (L/h=10). As for slope results, the maximum slope at the tip showed similar trend as the maximum deflection. For instance, the maximum slope at the tip for the beam with (L/h=40) was -0.115 rad compared to -0.0146 rad for the beam with (L/h=20).

Regarding interaction forces, results showed that as L/h ratio increases, both bending moment and shear force increase. According to Figure 4-9, the bending moment at support had a value of -0.087 N.m for the beam with (L/h=40) and -0.0054 N.m for the beam with (L/h=10). This trend was similar to the one for shear force at support, where the beam with (L/h=40) had maximum shear of 0.043 N, while the beam with (L/h=10) had a maximum shear of 0.011 N as shown in Figure 4-10.

As for stress and strain results, similar trend was realized for the cantilevered beams as the simply supported beams. It was shown that as the span – depth ratio increased, the maximum stress and strain values at top and bottom surfaces increased as shown in Figures 4-11 and 4-12.

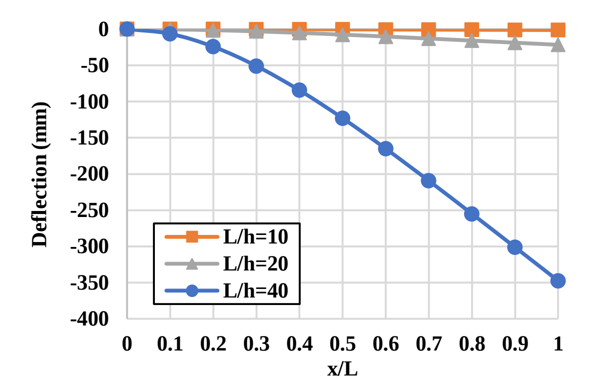


Figure 4- 7 : Deflection for fix supported beam with different L/h ratios.

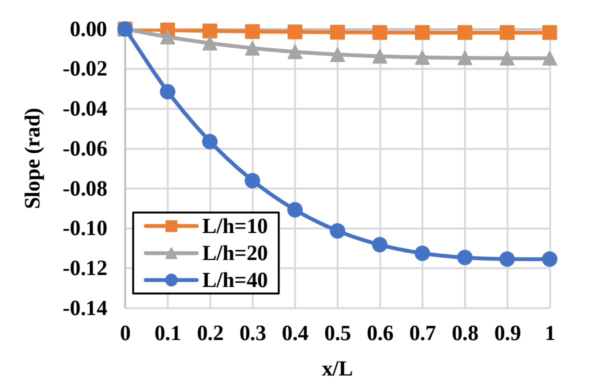


Figure 4- 8: Slope for fix supported beam with different L/h ratios.

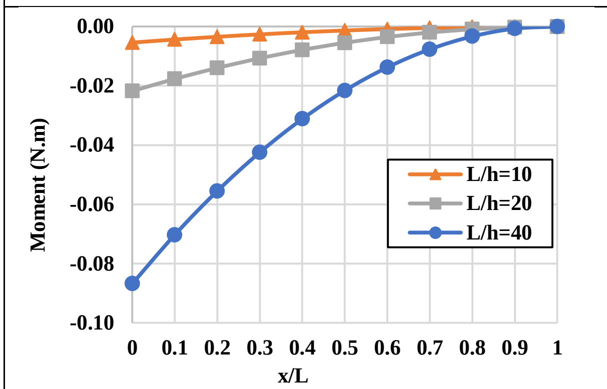


Figure 4- 9: Bending moment for fix supported beam with different L/h ratios.

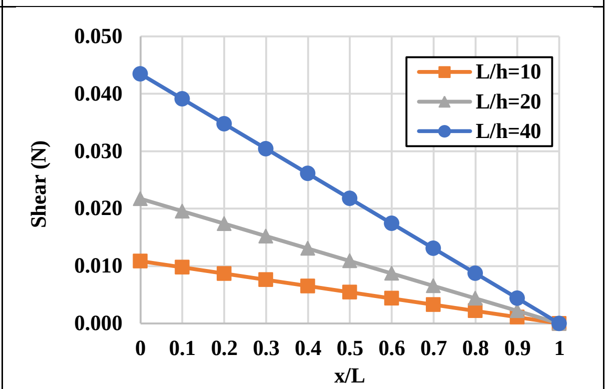


Figure 4- 10: Shear for fix supported beam with different L/h ratios.

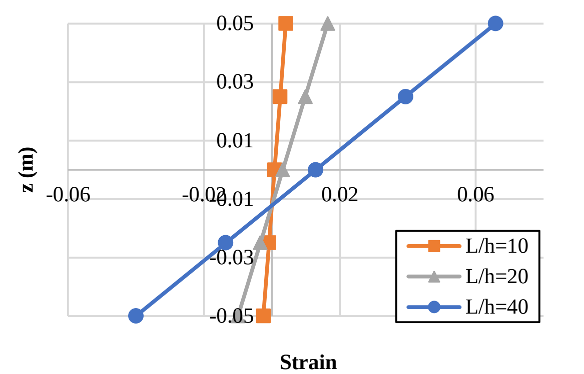
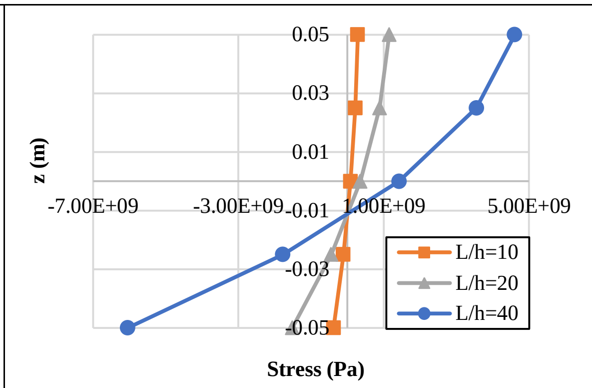
	

Figure 4- 11: Strain distribution for cantilevered beam with different L/h ratios.	Figure 4- 12: stress distribution for cantilevered beam with different L/h ratios.

The approximate solution suggested for FG beams mentioned in the study conducted by Karamanli (2016) proved to be close to the analytical solution. Figures 4-13 to 4-16 depict the stress distribution along the height of FG beam in Karamanli's study compared to the suggested approximate solution. It can be noted that the approximate solution result in closer stress values to the analytical solution. This was proved for both simply supported and cantilevered beams with different span – depth ratio as shown in Figures 4-13 to 4-16.

As for deflection results, the stiffness coefficient "Dxx" was determined for both analytical and approximate solution cases. It was shown that Dxx was close in both cases; 9.210×10^6 by analytical method and 9.207×10^6 by approximate method. Having these values to be too close means that the deflection will be similar in both cases since deflection equation depends on the coefficient Dxx as shown below:

$$\delta(x) = \frac{q_0 L^4}{24 Dxx}\left(\frac{x}{L} - \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right) \rightarrow \text{for simply supported} \qquad (4\text{-}1)$$

$$\delta(x) = \frac{q_0 L^4}{24 Dxx}\left(\frac{6x^2}{L^2} - \frac{4x^3}{L^3} + \frac{x^4}{L^4}\right) \rightarrow \text{for Cantilever} \qquad (4\text{-}2)$$

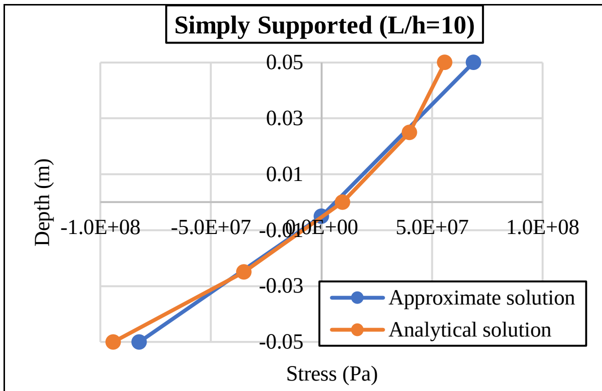


Figure 4- 13: Stress distribution for simply supported FG beam in Karamanli's study with (L/h=10) using analytical and approximate solutions.

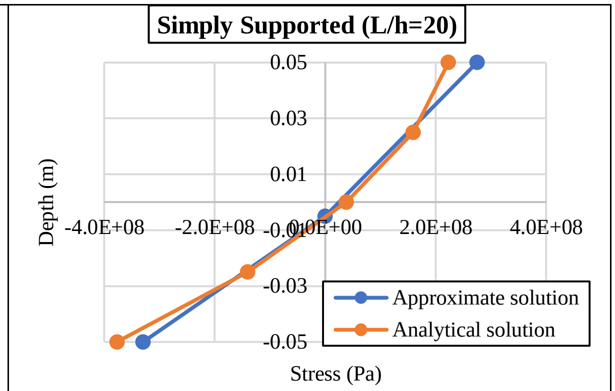


Figure 4- 14: Stress distribution for simply supported FG beam in Karamanli's study with (L/h=20) using analytical and approximate solutions

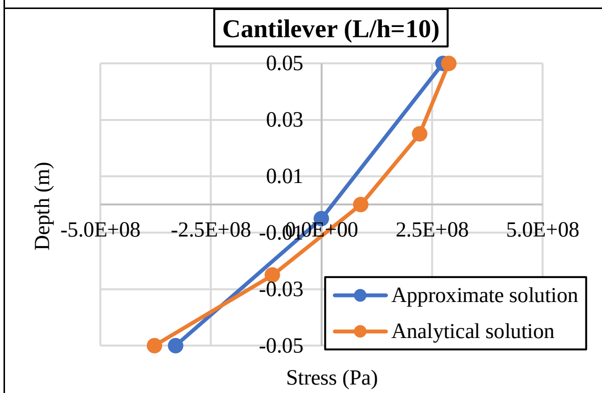


Figure 4- 15: Stress distribution for cantilever FG beam in Karamanli's study with (L/h=10) using analytical and approximate solutions

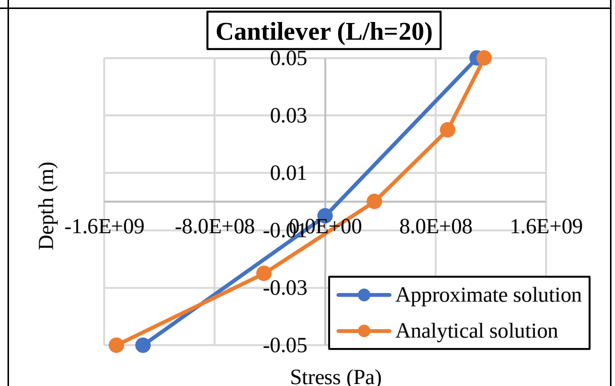


Figure 4- 16: Stress distribution for cantilever FG beam in Karamanli's study with (L/h=20) using analytical and approximate solutions

4.2.3 Effect of Material Function Used

Three functions were adopted to determine the response of FGB: power function, exponential function, and sigmoidal function. Table 4-2 summaries the maximum displacement results determined by analytical and approximate methods for both simply supported and cantilevered beams with (L/h=20). It can be clearly seen that both methods showed similar results with slight differences (less than 1%) regardless the function used in describing material gradation and for different boundary conditions. This demonstrates the validity of the suggested approximate method to be used in determining different beam results.

Table 4- 2: Beam results for different functions used

Function	Cantilever beam		Simply supported beam	
	$\delta_{max-Analytical}$ [a]	$\delta_{max-Approximated}$ [a]	$\delta_{max-Analytical}$ [a]	$\delta_{max-Approximated}$ [a]
Power	21.716	21.933	2.262	2.283
Exponential	21.692	22.126	2.260	2.284
Sigmoidal	21.707	21.967	2.260	2.283

a: maximum displacement (mm)

4.2.4 Effect of Power Value (k)

Figures 4-17 to 4-20 show the effect of different power values on the stress and strain distribution in both simply supported beams and cantilevered beams with (L/h=20). For a power value of 0, the material will be isotropic and has an equivalent elasticity modulus of (E1) as defined earlier. As the power value increases, the concentration of E1 material decreases and the concentration of E2 material increases. This will lead to a stiff behavior for the beam. For example, for k=0, the beam had higher stress compared to the case where k=1 (equal distribution of materials). This was shown for both simply supported and cantilever beam. As the power value increases beyond 1, the maximum stress value decreased since the concentration of E2 material is higher than E1 material (E2 > E1). This was reflected also on the deflection results shown in Figure 4-21, where the displacement for the beam with power value =10 was less than the beam with power values 0 and 1.

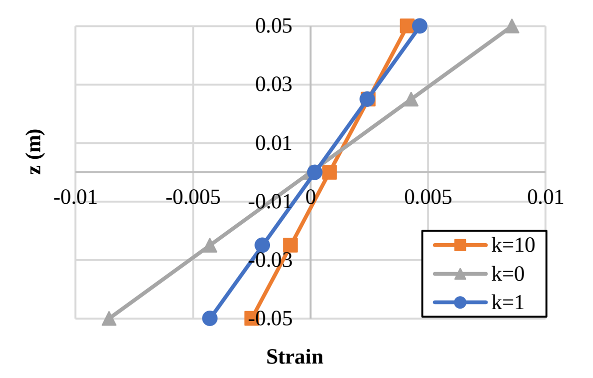


Figure 4- 17: Strain distribution for simply supported beam with different power values.

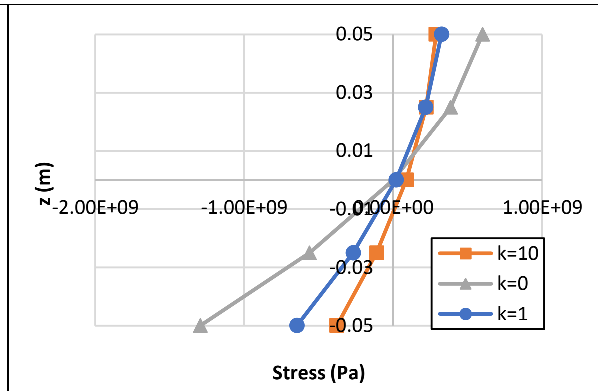


Figure 4- 18: stress distribution for simply supported beam with different power values.

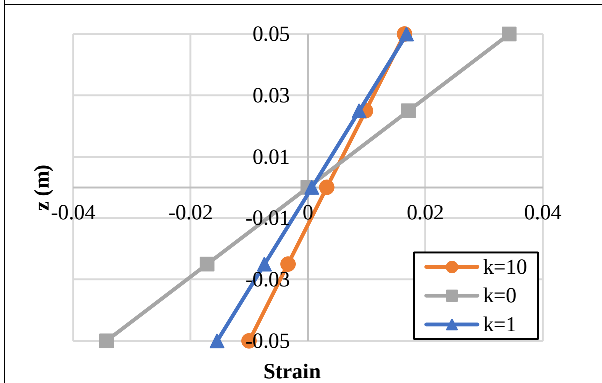


Figure 4- 19: Strain distribution for cantilevered beam with different power values.

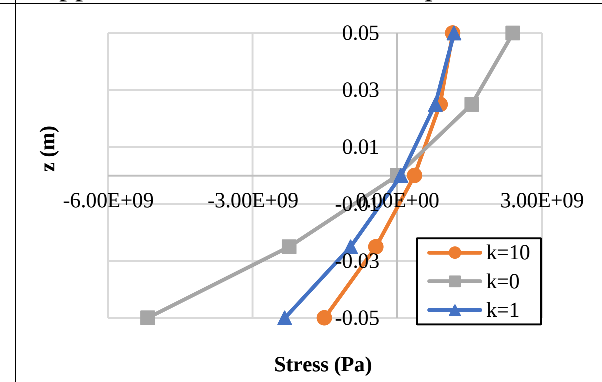


Figure 4- 20: stress distribution for cantilevered beam with different power values.

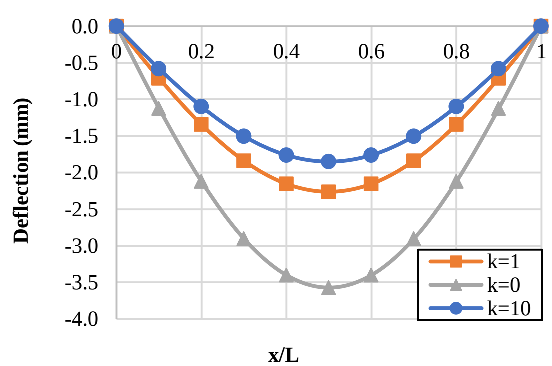


Figure 4- 21: Deflection for simply supported beam with different power values.

Regarding approximate method results, Table 4-3 summarizes the maximum displacement for FG beams with different power values using both analytical and approximate methods. It can be noted that the suggested approximate method leads to maximum displacement values that are too close to those determined by the analytical method. This again proves the validity of the suggested approximate methods to be applied for FG bending problems

Table 4- 3: Approximate vs analytical maximum displacement results for k=0, 1, and 10

Function	Simply supported beam	
	$\delta_{max-Analytical}$ [a]	$\delta_{max-Approximated}$ [a]
k=1	2.262	2.283
k=0	3.571	3.642
k=10	1.850	1.876

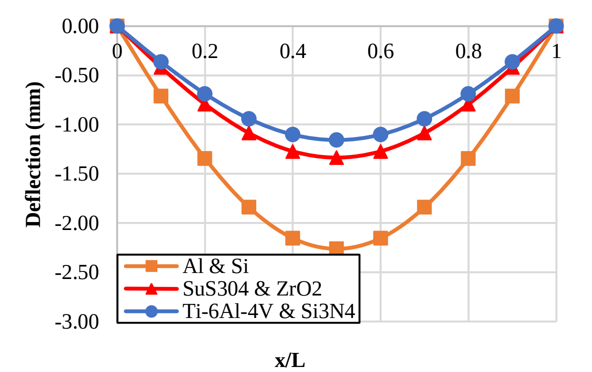
a: maximum displacement (mm)

4.2.5 Effect of Different Material Combination

Another material combination was considered to evaluate the behavior of FGM beams with different material combinations. The material parameters were taken from the research conducted by Kadoli et al (2008). The materials were Zirconia (ZrO_2), Metal (SuS304), Titanium alloy (Ti–6Al–4V), and Silicon Nitride (Si_3N_4). These materials have different properties than the materials used before in this study (Karamanli (2016)).

Figures 4-22 to 4-27 summarizes the main results for this new material combination compared to the previous one for simply supported beams. In terms of deflection, results showed that new combinations showed less deflection than the one adopted by Karamanli (2016) (Al & Si). This is because the new materials have higher elasticity modulus than the previous ones ($E_{(ZrO2)}$ = 168 GPa, $E_{(SuS304)}$ = 208 GPa, $E_{(Si3N4)}$ = 322 GPa, and $E_{(Ti\text{-}6Al\text{-}4V)}$ = 110 GPa) which means less defection. Similar trend was realized for other findings like slope (Figure 4-23), bending moment (Figure 4-24), shear force (Figure 4-25), strain

(Figure 4-26) and stress (Figure 4-27). The reduction percentage reached 48.8% for maximum deflection, maximum slope, maximum bending moment, and maximum shear. Similarly, for stress and strain values, they were 30%-35% less than the previous material combination used.



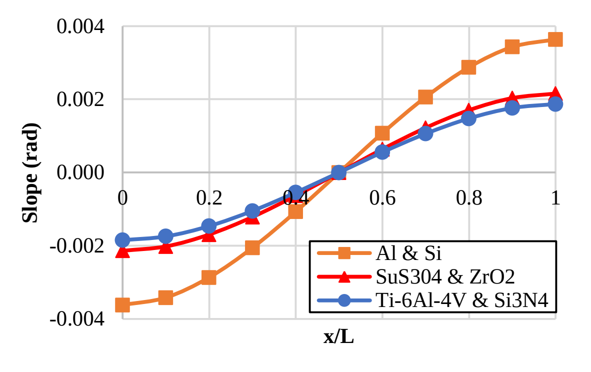
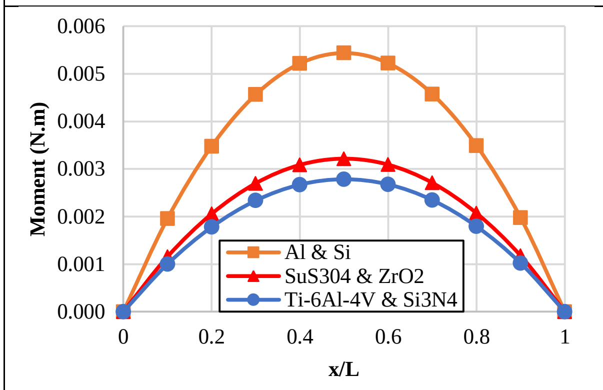


Figure 4-22: Deflection for simply supported beam with different material combinations.

Figure 4-23: Slope for simply supported beam with different material combinations.



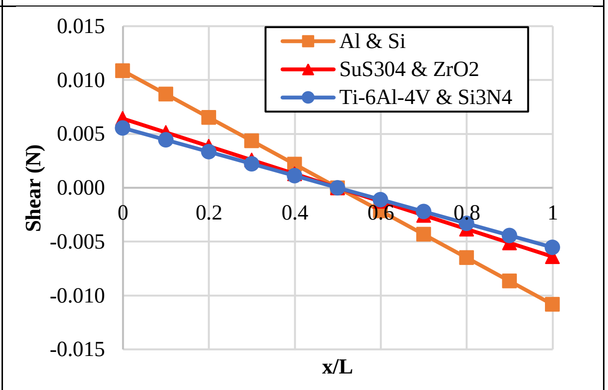


Figure 4-24: Bending moment for simply supported beam with different material combinations.

Figure 4-25: Shear for simply supported beam with different material combinations.

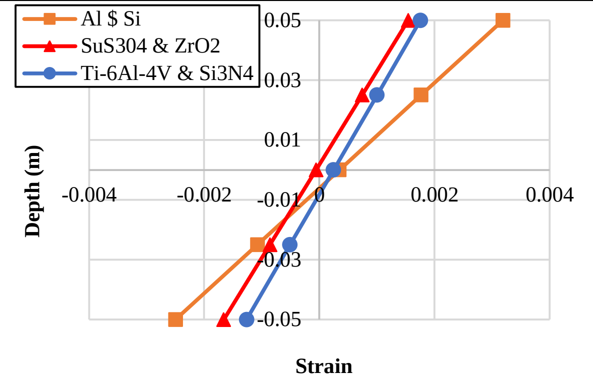


Figure 4- 206: Strain for simply supported beam with different material combinations.

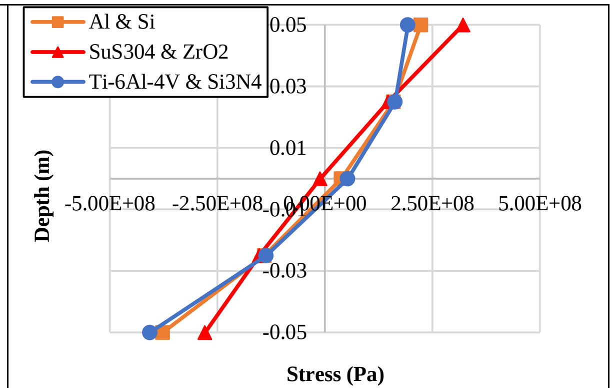


Figure 4- 217: Stress for simply supported beam with different material combinations.

As for the approximate method suggested for the problem, Figures 4-28 to 4-31 summarizes strain and stress distribution results for FG beams with SuS304-ZrO2 and Ti-6Al-4V-Si3N4. Both simply supported and cantilever beams were considered. It can be noted that the approximate solution led a close strain and stress distribution to the analytical solution. The difference between the maximum stresses was trivial and reached 1% in simply supported cases.

As for deflection curves, both methods led to similar curves since the stiffness constant Dxx from approximate solution was close to that from analytical solution. For instance, Dxx for FG beam with SuS304-ZrO2 was $1.56 \times 10^7 N.m^2$ from the analytical solution and $1.57 \times 10^7 N.m^2$from the approximate solution. As for FG beam with Ti-6Al-4V-Si3N4, Dxx was $1.80 \times 10^7 N.m^2$ from the analytical solution and $1.87 \times 10^7 N.m^2$from the approximate solution. Again, this proves the validity of the suggested approximate solution for FG beams.

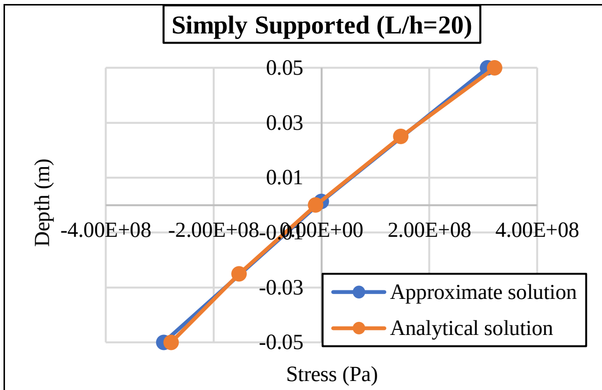


Figure 4- 228: Approximate vs analytical stress distribution for SuS304-ZrO2 FG simply supported beam.

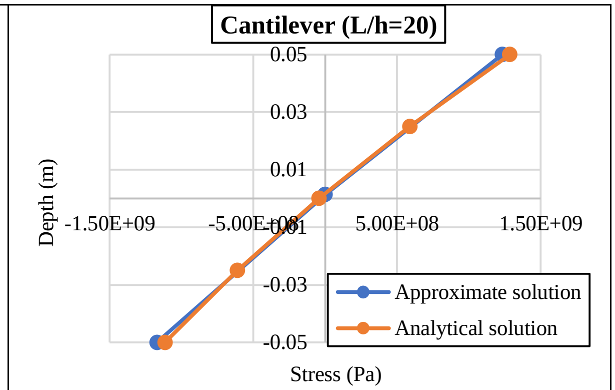


Figure 4- 239: Approximate vs analytical stress distribution for SuS304-ZrO2 FG Cantilever beam.

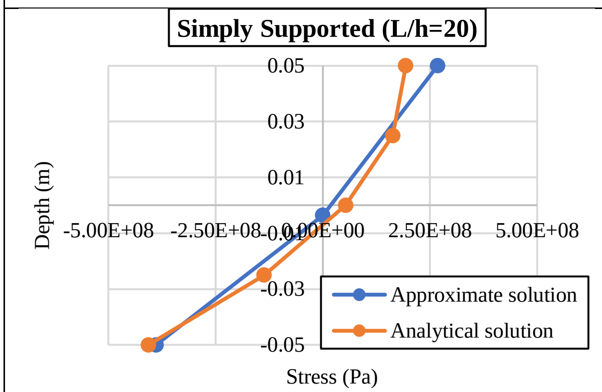


Figure 4- 30: Approximate vs analytical stress distribution for Ti-6Al-4V-Si3N4 FG simply supported beam.

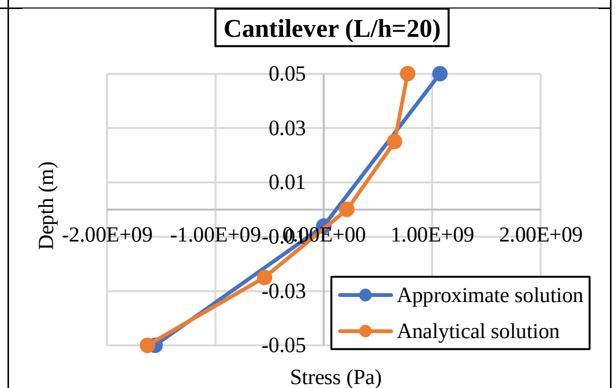


Figure 4- 31: Approximate vs analytical stress distribution for Ti-6Al-4V-Si3N4 FG Cantilever beam.

4.3 Functionally Graded Columns

As mentioned earlier in chapter three, two column cases were considered in the buckling analysis; the simply supported column and the cantilever column. Both cases were studied at three span – depth ratios: 10, 20, and 40. The compression load considered in this study was 1 MN. In addition, a comparison was done on the FG material models (power, exponential, and sigmoidal) to check the difference in the results among them.

Furthermore, approximate solutions were compared to analytical solutions to confirm its validity for FG column buckling problems.

4.3.1 Effect of Span – Depth (L/h) Ratio

Three Span to depth (L/h) ratios were considered in this study: 10, 20 and 40. Figure 4-32 depicts the displacement diagram at each distance ratio (x/L) measured for the simply supported column. It can be seen that as span – depth ratio increases, the deflection of simply supported column at any point increases. For example, the maximum displacement for the column with (L/h=40) was 1.62 mm, while for (L/h=10) the maximum deflection was 0.09 mm.

As for the critical buckling load, it was shown that as the span – depth ratio increased, the buckling load decreased as depicted in Figure 4-33. The load decreased from 90 MN for L/h=10 to 5.7 MN for L/h=40.

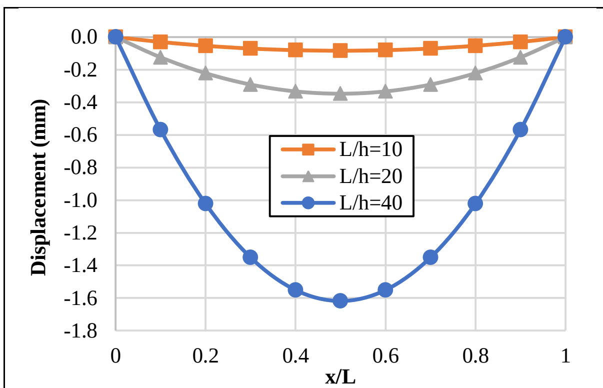


Figure 4-32: lateral displacement for simply supported columns with different L/h ratios.

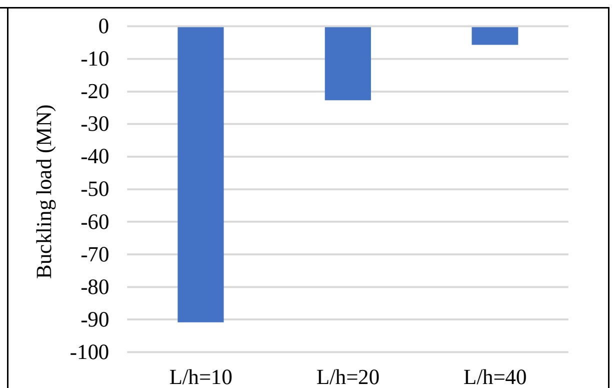


Figure 4-33: Buckling load for simply supported columns with different L/h ratios.

Figure 4-34 depicts the displacement diagram at each distance ratio (x/L) measured for

the cantilevered column. It can be seen that as span – depth ratio increases, the deflection of cantilevered column at any point increases. For example, the maximum displacement for the column with (L/h=40) was 18.3 mm, while for (L/h=10) the maximum deflection was 0.35 mm.

As for the critical buckling load, it was shown that as the span – depth ratio increased, the buckling load decreased as depicted in Figure 4-35. The load decreased from 22.7 MN for L/h=10 to 1.4 MN for L/h=40.

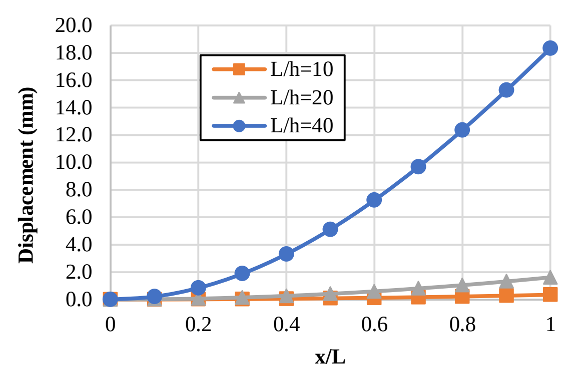


Figure 4-34: lateral displacement for cantilevered columns with different L/h ratios.

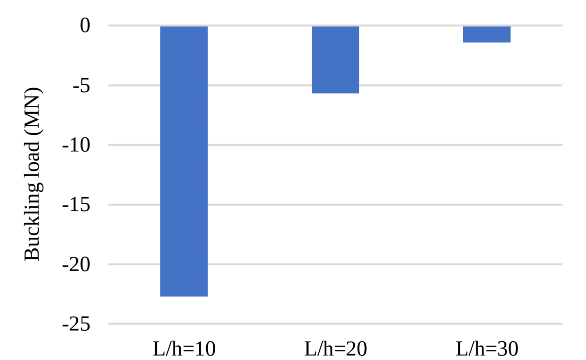


Figure 4-35: Buckling load for cantilevered columns with different L/h ratios.

As for approximate methods adopted to determine the buckling load for FG columns, Figures 4-36 to 4-39 show the critical buckling load determined by Analytical method, Rayleigh method, Timoshenko method, and Rayleigh-Ritz method. It can be depicted that for both simply supported and cantilever columns, Rayleigh-Ritz method resulted in the most accurate critical buckling load compared to the analytical method. In average, Rayleigh quotient method led to an error percentage of 22% for simply supported column and 1.5% for cantilever column compared to the analytical solution. As for Timoshenko quotient method, the error percentage was 1.4% for simply supported columns and 0.8% for cantilevered columns. Finally, for Rayleigh-Ritz method, the error percentage was the least among all approximate methods and was around 0.7% for both boundary conditions.

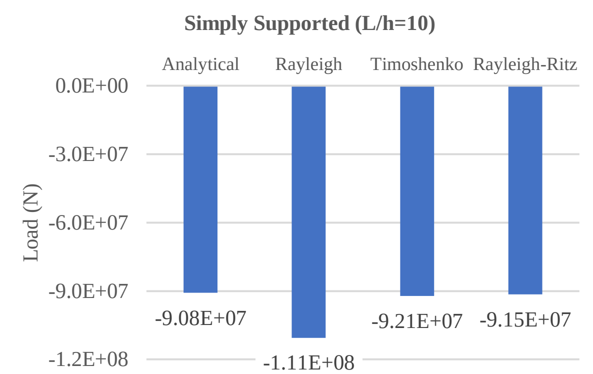


Figure 4-36: Buckling load for simply supported columns with (L/h=10) using different approximate method solutions.

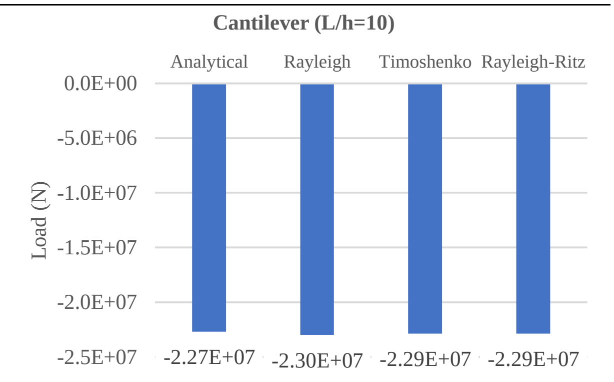


Figure 4-37: Buckling load for cantilever columns with (L/h=10) using different approximate method solutions.

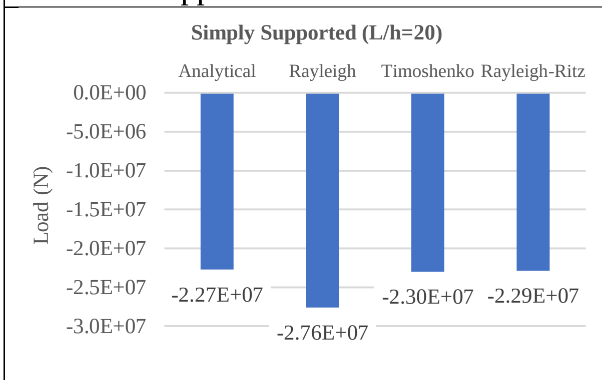


Figure 4-38: Buckling load for simply supported columns with (L/h=20) using different approximate method solutions.

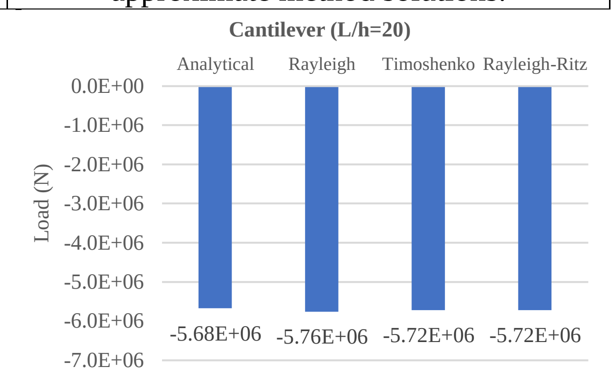


Figure 4-39: Buckling load for cantilever columns with (L/h=20) using different approximate method solutions.

4.3.2 Effect of Material Function Used

Three functions were adopted to determine the response of FGB: power function, exponential function, and sigmoidal function. Table 4-4 summaries the results for both simply supported columns and cantilevered columns with (L/h=20). It can be clearly seen that all methods showed similar results with slight differences (less than 1%) for both simply supported and cantilevered columns. This demonstrates the validity for these methods to be used in determining different column results.

Table 4- 3: Column results for different material functions used

Function	Cantilever column		Simply supported column	
	P_{cr}	δ_{max}	P_{cr}	δ_{max}
Power	5675432.25	1.618021769	22701729	-0.347324583
Exponential	5681594.50	1.615883463	22726378	-0.346930085
Sigmoidal	5678513.38	1.616951909	22714053.5	-0.347127222

The same columns were analyzed by approximate solution methods with different materials functions. Table 4-5 shows that all material functions resulted in similar buckling loads for FG columns with error less that 1%. This proved the validity of different materials functions in approximate solution methods for buckling problems.

Table 4-5: Critical buckling load using approximate methods for different material functions

Function	P_{cr} - Cantilever column (N)			P_{cr} - Simply supported column (N)		
	Rayleigh	Timoshenko	Rayleigh-Ritz	Rayleigh	Timoshenko	Rayleigh-Ritz
Power	5.76E+06	5.72E+06	5.72E+06	2.76E+07	2.30E+07	2.29E+07
Exponentia l	5.78E+06	5.73E+06	5.73E+06	2.78E+07	2.32E+07	2.30E+07
Sigmoidal	5.77E+06	5.73E+06	5.72E+06	2.77E+07	2.30E+07	2.31E+07

4.3.3 Effect of Power Value (k)

The power value demonstrates the distribution of materials among the column length. For power constant = 0, the material will be isotropic with E=E1. As for power =1, both materials will have homogeneous distribution (material 1 will have 100% intensity at the bottom of the column and reduces gradually until reaching 0% at the top of the column). Figures 4-40 and 4-41 summarizes lateral displacement and critical buckling load values for simply supported columns with power values =0, 1, and 10.

According to Figure 4-40, as the power (k) increases, the lateral displacement decreases. This is because the material with E2 will have higher concentration in the column, and E2 > E1, which means the column will have higher stiffness since the (EI) will be greater as the value of k increases. This will lead to a less lateral displacement for the same applied compression force. This, in the other hand, means that the critical buckling load will increase as the power value increases since the column now had higher stiffness.

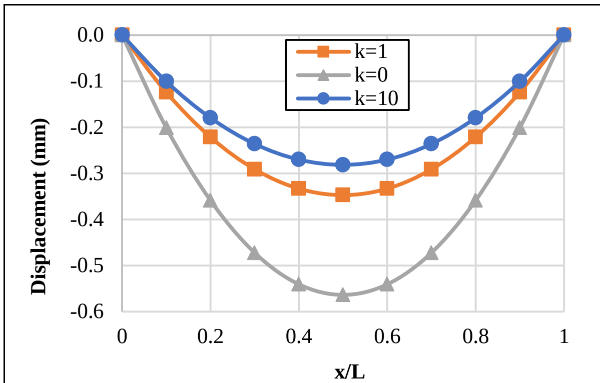


Figure 4- 40: lateral displacement for simply supported columns with different power values.

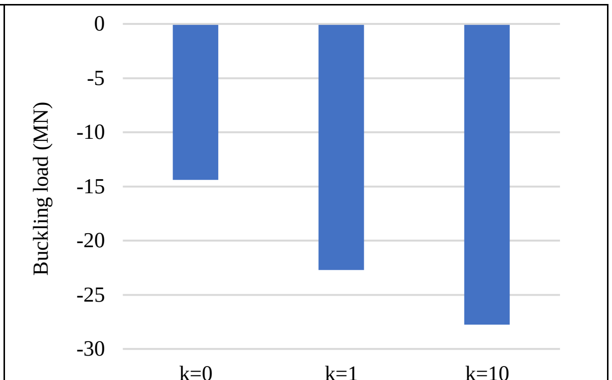


Figure 4- 41: Buckling load for simply supported columns with different power values.

Approximate solution methods were applied for FG columns with different power values. Table 4-6 summarizes the critical buckling load for FG columns with power values = 0, 1, and 10. It can be noted that approximate method results were close to analytical method ones. In addition, the Rayleigh – Ritz method was the most accurate method compared to other approximate solution methods with error percentage of approximately 0.75%.

Table 4-6: Critical buckling load using approximate methods for different power values

Function	P_{cr} - Simply supported column (N)			
	Analytical	Rayleigh	Timoshenko	Rayleigh-Ritz
k=0	1.44E+07	1.76E+07	1.46E+07	1.45E+07
k=1	2.27E+07	2.76E+07	2.30E+07	2.29E+07
k=10	2.78E+07	3.34E+07	2.80E+07	2.79E+07

4.3.4 Effect of Different Material Combination

Two other material combinations were investigated to validate approximate methods suggested by the author. The material combinations are metal (SuS304) – ceramic (ZrO_2), and Titanium (Ti-6Al-4V) – silicon nitride (Si_3N_4). Figures 4-42 to 4-45 summarizes the critical buckling load for FG column with span depth ratio = 20 using analytical method, Rayleigh quotient, Timoshenko quotient, and Rayleigh-Ritz quotient. It can be claimed that approximate solutions resulted a close solution to the analytical solution. Also, it was shown that Rayleigh – Ritz quotient was the best method to determine critical buckling load with error percentage less than 0.7%.

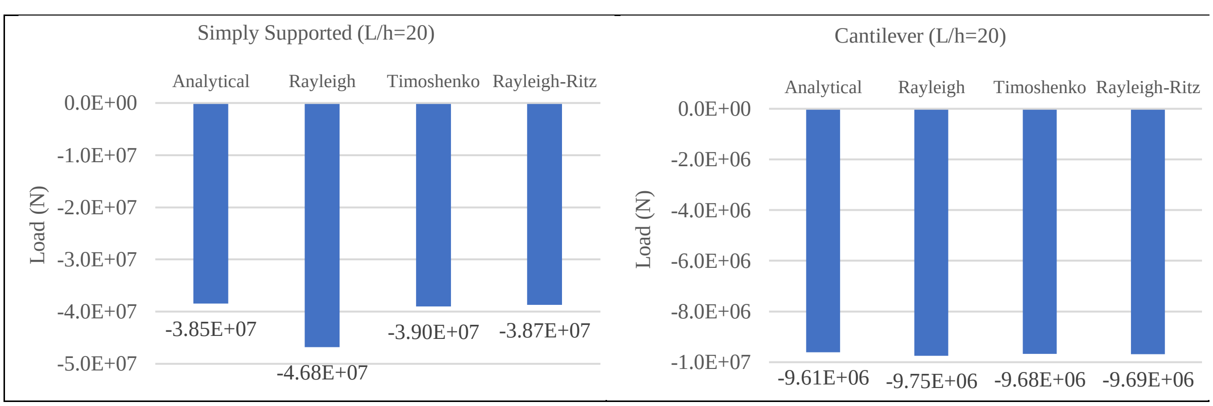


Figure 4- 42: Buckling load for simply supported column with SuS304-ZrO_2 material combination

Figure 4- 43: Buckling load for cantilever column with SuS304-ZrO_2 material combination.

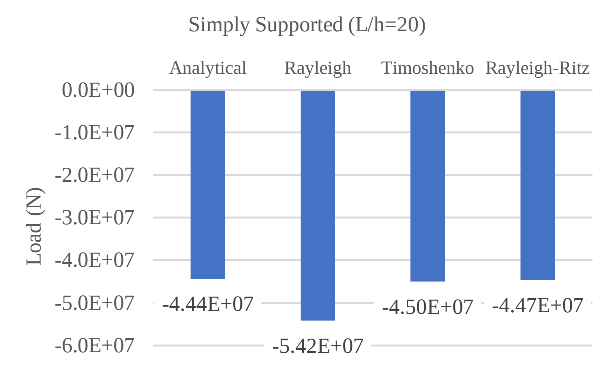


Figure 4- 44: Buckling load for simply supported column with Ti-6Al-4V-Si_3N_4 material combination.

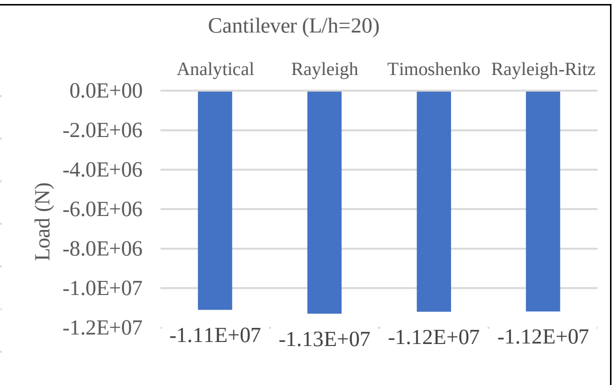


Figure 4- 45: Buckling load for cantilever column with Ti-6Al-4V-Si_3N_4 material combination.

4.4 Chapter Summary

This chapter showed the analysis results for functionally graded beams and columns with different boundary conditions, different material concentrations, different geometrical properties, and different material combinations using both analytical and approximate methods. The approximate methods adopted were Rayleigh quotient, Timoshenko quotient, and Rayleigh-Ritz quotient. It was proved that approximate solution led to closer results to the analytical solution. In addition, it can be claimed that the best approximate method used to analyze FG columns is Rayleigh-Ritz quotient.

Chapter Five: Conclusions and Recommendations

5.1 Aim and Introduction

The last chapter will exhibit the main conclusions of this research work along with future work recommendations. This will be important for any researcher who would like to continue in the same area to cover any gaps or missing outputs, or to add to the current results and findings.

5.2 Conclusions

This work conducts a modeling and analysis of FG beams and columns to compare between approximate methods and analytical methods to determine different outputs related to bending and buckling problems. The results of this work can be concluded as:

1- The approximate solution suggested for bending of FG beams proved to work efficiently. The method was tried for different boundary conditions (simply supported and fixed) and with different combination of materials. Different outputs were included in the comparison between the two methods like: bending moment, shear force, deflection, slope, stress distribution, and strain distribution. The error in all outs puts was trivial (~0%) and proved the validity of the approximate method suggested for FG beams.

2- It was proved by both approximate and analytical solution methods that using high stiffness materials in FG beams enhanced its structural behavior. This was shown by comparing maximum deflection and maximum moment for FG beams with different material combination, where beams with higher elasticity modulus materials had less maximum deflection and maximum moment compared to beams with less elasticity modulus materials.

3- The approximate solutions derived for buckling analysis of FG columns proved to work efficiently. This was checked by comparing the critical buckling load of different FG columns with different boundary and materials conditions. Among all approximate methods tried in this work, the Rayleigh – Ritz method proved to be the most accurate one with error percentage less than 0.75%. On the other side, Rayleigh methods seemed to be the less accurate one with error percentage of 22%.

4- It was proved by both approximate and analytical solution methods that using high stiffness materials in FG columns increased its critical buckling load, where FG columns with higher elasticity modulus materials had critical buckling load compared to columns with less elasticity modulus materials.

5- Approximate solutions proved to be effective in saving time and effort in analysis, as well as, it is good enough for practical purposes.

5.3 Future Work

Based on the findings of this research, the researcher proposes a set of suggestions for future work, to engineers, scholars and scientists, who are willing to conduct further investigations on the key role of FGMs in strengthening and improving the performance of columns and beams. The future work suggestions are:

A) Developing approximate solutions for dynamic problems in both FG beams and columns

B) Developing approximate solutions for other structural member problems like FG walls and slabs

C) Modeling FG beams and columns on finite element software to extract different complex outputs.

D) Further research take advantage of this research codes in order to determine certain parameters for designing the FGM.

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Appendix 1

Appendix A: MATLAB CODES for different Function cases

A.1. Power function

```
clc;
clear all;
h=0.1;
l=1;
b=0.1;
q=100000;
E1=7*10^10;
E2=1.51*10^11;
k=1;
delta=0.0001;
Mc=(q*l.^2)/8;
Ez= @(z) (E2+(E1-E2)*((z/h)+0.5).^k);
zn=0.3*h;
zh=0;
for ii=1:10
fE1=(quad(Ez,h/2-zn,h/2))/zn;
fE2=(quad(Ez,-h/2,h/2-zn))/(h-zn);
zh=(((fE2)^0.5)*h)/((((fE2)^0.5)+(((fE2)^0.5))));
err=((zh-zn)/zn);
err
ii=ii+1;
if abs(zh-zn)<= delta
break;
else
zn=zh;
end
end
eps1=(3*Mc)/(fE1*b*h*zn);
eps2=((h-zn)*eps1)/zn;
stress1=E1*eps1;
stress2=E2*eps2;
```

A.2. Exponential function

```
clc;
clear all;
h=0.1;
l=1;
b=0.1;
q=100000;
E1=7*10^10;
E2=1.51*10^11;
k=1;
delta=0.0001;
zn=0.05;
Mc=(q*l.^2)/8;
alpha=log(E1/E2);
Ez= @(z) (E2*exp(alpha*(0.5+(z/h)).^k));
zh=0;
for ii=1:10
fE1=(E1*h*(1-(1/(E1/E2)^(zn/h)))/(zn*alpha));
fE2=(E2*h*(-1+((E1/E2)^(1-zn/h)))/((h-zn)*alpha));
zh=(((fE2)^0.5)*h)/((((fE1)^0.5)+(((fE2)^0.5)))); zh
err=((zh-zn)/zn);
%err
ii=ii+1;
if abs(zh-zn)>= delta
zn=zh;
else
break;
end
end
eps1=(3*Mc)/(fE1*b*h*zn);
eps2=((h-zn)*eps1)/zn;
stress1=fE1*eps1;
stress2=fE2*eps2;
c=0.5*fE1*eps1*zn*b;
T=0.5*fE2*eps2*(h-zn)*b;
Mmax=(2/3)*h*c;
```

A.3. Sigmoidal Function

```
clc;
clear all;
format long ;
h=0.1;
l=1;
```

```
b=0.1;
q=10000;
E1=7*10^10;
E2=1.51*10^11;
k=1;
delta=0.0001;
Mc=(q*l.^2)/8;
E1z= @(z) (E2+(E1-E2)*((z/h)+0.5).^k);
E2z= @(z) (E2+(E1-E2)*((z/h)-0.5).^k);
zn=0.3*h;
zh=0;
for ii=1:2
l1=h/2-zn;
l2= h/2;
l3=-h/2;
l4=h/2-zn;
if l1>0 && l2>0
fE1=(quad(E1z,l1,l2))/zn;
else
fE1=(quad(E2z,l1,l2))/zn;
end
if l1<0 && l2>0
fE1=((quad(E2z,l1,0))+(quad(E1z,0,l2)))/zn;
end
if l3<0 && l4<0
fE2=(quad(E1z,l3,l4))/(h-zn);
end
if l3<0 && l4>0
fE2=((quad(E2z,l3,0))+(quad(E1z,0,l4)))/(h-zn);
end
zh=(((fE2)^0.5)*h)/((((fE1)^0.5)+(((fE2)^0.5))));
zn=zh;
zn
err=((zh-zn)/zn);
ii=ii+1;
if abs(zh-zn)>= delta
zn=zh;
else
break;
end
end
eps1=(3*Mc)/(fE1*b*h*zn);
eps2=((h-zn)*eps1)/zn;
stress1=fE1*eps1;
stress2=fE2*eps2;
c=0.5*fE1*eps1*zn*b;
```

T=0.5*fE2*eps2*(h- zn)*b;
Mmax=(2/3)*h*c;

حلول التقريبية لإنحناء الجيزان وانبعاج الاعمدة المصنوعة من مواد متدرجة وظيفياً

أعدت من قبل
مصعب حسن أحمد الطعاني

أشرف عليها
أ.د. كريم سويلم نمير

الملخص

في هذا البحث تم تطوير حلول تقريبية لحل مشاكل الجيزان والاعمدة للمواد المتدرجة وظيفياً. يعتبر المواد المتدرجة وظيفيا في الاتجاه العرضي لكل من الجيزان والأعمدة باستخدام وظائف مختلفة؛ دالة الطاقة، الدالة الأسية، والدالة السينية. أيضًا، تُستخدم قيم طاقة مختلفة للنظر في تركيزات مختلفة للمواد في الاتجاه العرض للجيزان والأعمدة. تم تطوير الحلول التقريبية لأول مرة لحزم المواد المتدرجة وظيفيا باستخدام تقنيات تكرارية تعتمد على حساب متوسط معامل المرونة في شد الجيزان ومناطق الانضغاط لتحديد النواتج الهيكلية المختلفة مثل مخططات الانحراف والانحدار والقص والعزم. ثم تتم مقارنة المخرجات بنظرية انحناء أويلر الكلاسيكية للتحقق من صحة الطريقة التقريبية المعتمدة المستخدمة في الحل. بالإضافة إلى مشاكل انحناء الجيزان، تم أيضًا تطوير حلول تقريبية لمشاكل انبعاج الاعمدة للمولد المتدرجة وظيفيا. الطرق المعتمدة في الحل هي حاصل رايلي وحاصل تيموشينكو وطريقة رايلي- ريتز. تتم مقارنة هذه الطرق بحلول أويلر التحليلي الكلاسيكي. تم اعتماد برنامج ماتلاب لحل كلتا المشكلتين لأنه بسيط ودقيق. سلطت النتائج الرئيسية لهذه الدراسة الضوء على أهمية الحلول التقريبية في حل مشاكل الانحناء وانبعاج المتدرجة وظيفيا. بالنسبة لمشاكل

الانحناء، نتج عن الطريقة التقريبية المقترحة خطأ طفيف (~ 0٪) عند مقارنتها بالحل التحليلي. أما بالنسبة لمشاكل انبعاج، فقد أظهرت طريقة رايلي-ريتز أنها الأكثر دقة في حساب حمل الانبعاج الحرج مع خطأ أقل من 0.75٪، بينما أدت طريقة رايلي إلى أخطاء كبيرة (~ 22٪). في نهاية هذا العمل، تم تقديم بعض التوصيات للعمل البحثي المستقبلي على التدرج الوظيفي للهياكل.

الكلمات الرئيسية: مواد متدرجة وظيفيًا، لإنحناء الجيزان، انبعاج الأعمدة، الحل التقريبي.