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# Economic-Statistical Design of Multivariate Control Charts Using Quality Loss Function

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*When a control chart is applied to monitor a production process, three test parameters should be determined: the sample size, the sampling interval between successive samples, and the control limits or critical region of the chart. In this paper, we develop the procedure to carry out the economic-statistical design of multivariate control charts by using a quality loss function for monitoring the process mean vector and covariance matrix simultaneously; i.e., to determine economically the optimum values of the three test parameters so that the statistical constraints (including the requirements of type I error probability and power) of the control chart can be satisfied. The test statistic $-2\ell nL$ is used to develop this procedure and the cost model is established based on the cost function developed by Montgomery and Klatt and the multivariate quality loss function presented by Kapur and Cho. A numerical example is provided to illustrate the solution procedure of the design and then the effects of cost parameters on the optimal design are studied.*

**Keywords:** Control chart; Economic–statistical design; Multivariate quality loss function; Statistical process control

## 1. Introduction

Statistical process control is an effective approach for improving product quality and saving production costs for a process. Since 1924 when Dr Shewhart presented the first control chart, various control-chart techniques have been developed and widely applied as a primary tool in statistical process control. The major function of control-charting is to detect the occurrence of assignable causes, so that the necessary corrective action can be taken before a large quantity of nonconforming product is manufactured. The control-charting technique may be considered as the graphical expression and operation of statistical hypothesis testing. When a control chart is used to monitor a process, three test parameters should be determined: the sample size, the sampling interval between successive samples, and the control limits or critical region of the chart.

Duncan [1] proposed the first economic model for determining the three test parameters for the *X*-bar control chart that minimises the average cost when a single out-of-control state (assignable cause) exists. Duncan's cost model includes the cost of sampling and inspection, the cost of defective products, the cost of a false alarm, the cost of searching for an assignable cause, and the cost of process correction. Since then, considerable attention has been devoted to the optimal economic determination of the three parameters of *X*-bar charts [2–5]. Montgomery [6] gave a thorough review of the literature of the economic designs of various control charts. A bibliography of related papers is also available from Vance [7] and Ho and Case [8]. Alexander et al. [9] combined Duncan's cost model with the Taguchi loss function to develop a loss model for determining the three test parameters. This loss model explicitly considers the quality loss due to process variability, which is not accounted for in Duncan's cost model. Since the solution from an economic design of control charts may have poor statistical properties, Saniga [10] presented the economic–statistical design for control charts in which the cost function is minimised subject to the constrained minimum value of power and maximum value of the type I error probability.

Although much work has been done on the economic designs of control charts that measure a single characteristic, some industrial products and processes are characterised by two or more measurable characteristics, and their joint effect describes product quality. For example, in the production of synthetic fibre, the tensile strength and diameter may be equally important quality characteristics. These characteristics are jointly distributed random variables and cannot appropriately be controlled by independently applying a control chart to each variable. Some authors [11–13] have developed quality control procedures for several related random variables. Among these procedures, the Hotelling $T^2$ control chart is probably the most widely known. Montgomery and Klatt [14] presented a cost

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model to economically design the $T^2$ control charts. Chen [15] conducted the economic–statistical design of the $T^2$ control charts by adding statistical constraints to the design procedure.

Since the $T^2$ control charts monitor the process mean vector only, and the process covariance matrix may also have an impact on product quality, in this paper we apply the test statistic $-2\ell nL$ (which will be reviewed in Section 3) to develop the economic–statistical design of multivariate control charts for monitoring the process mean vector and covariance matrix simultaneously. The cost function given in Montgomery and Klatt [14] will be combined with the multivariate loss function presented in Kapur and Cho [16] to develop a cost model that is used as the objective function of the design which is to be minimised. An example is provided to illustrate the solution procedure of the design and then some sensitivity analyses are conducted to investigate the effects of cost parameters on the solution of the design.

## 2. Model Assumptions

To simplify the mathematical manipulation and analysis of the control chart, the following assumptions are made.

1. The quality of the process can be described by the mean vector and covariance matrix of $p$ characteristics, and is monitored by a multivariate control chart using the test statistic $-2\ell nL$.
2. The $p$ quality characteristics monitored by the multivariate control chart are all nominal-the-best (i.e., a target vector exists) and follow a multivariate normal distribution with a mean vector $\mu$ and a covariance matrix $\Sigma$.
3. In the start of the process, the process is assumed to be in-control; that is, $\mu = \mu_0$ and $\Sigma = \Sigma_0$, where $\mu_0$ is a given vector and $\Sigma_0$ is a given positive definite matrix.
4. It is further assumed that the target vector is the same as $\mu_0$.
5. There are two out-of-control states caused by two assignable causes respectively. After the first assignable cause occurs, the process mean vector shifts to $\mu_1 = \mu_0 + \Delta_\mu$ and the process covariance matrix remains unchanged, where the $p \times 1$ vector $\Delta_\mu$ is known. After the second assignable cause occurs, the process covariance matrix shifts to $\Sigma_1 = \Sigma_0 + \Delta_\Sigma$ and the process mean vector remains unchanged, where the $p \times p$ matrix $\Delta_\Sigma$ is known.
6. The time the process remains in the in-control state before going out of control is assumed to follow an exponential distribution with a mean of $\lambda^{-1}$ hours.
7. When the process goes out of control, it stays out of control until detected and corrected.
8. During each sampling interval, there exists at most one assignable cause which makes the process out of control. The assignable cause will not occur at the sampling time.
9. When the multivariate control chart indicates the process is out of control, the process is stopped for investigating the assignable cause.
10. The cost for investigating real and false alarms is the same.
11. Considering the statistical properties of the multivariate control chart, the upper bound of the type I error probability is set to be 0.1 and the lower bounds of the powers for the two out-of-control states are all set to be 0.9.

## 3. The Test Statistic

Suppose that the output of a process can be described by $p$ quality characteristics, and $\mathbf{Y}$ is a $p \times 1$ random vector whose $j$th element (denoted by $y_j$) is the $j$th quality characteristic and is multivariate normally distributed. Let $E(\mathbf{Y}) = \mu$ be the $p \times 1$ mean vector of the characteristics and $Cov(\mathbf{Y}) = \Sigma$ be the $p \times p$ covariance matrix of $\mathbf{Y}$. Generally, $\mu$ and $\Sigma$ are unknown. For a random sample of size $n$ from $\mathbf{Y}$, say $\mathbf{Y_1}$, $\mathbf{Y_2}$, …, $\mathbf{Y_n}$, the sample mean vector and sample covariance matrix may be computed by

$$\overline{\mathbf{Y}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i \tag{1}$$

$$\mathbf{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{Y}_i - \overline{\mathbf{Y}})(\mathbf{Y}_i - \overline{\mathbf{Y}})^T \tag{2}$$

where superscript $T$ denotes the transpose operation. The likelihood ratio criterion of testing the hypothesis $H_0$: $\mu = \mu_0$ and $\Sigma = \Sigma_0$ against alternatives $H_1$: $\mu \neq \mu_0$ or $\Sigma \neq \Sigma_0$ may be expressed as [17]

$$L = \left(\frac{e}{n}\right)^{\frac{np}{2}} \left|(n-1)\mathbf{S}\Sigma_0^{-1}\right|^{\frac{n}{2}}$$

$$exp\left\{-\frac{1}{2}\,\mathrm{Tr}\,\Sigma_0^{-1}[(n-1)\mathbf{S} + n\,(\overline{\mathbf{Y}} - \mu_0)(\overline{\mathbf{Y}} - \mu_0)^T]\right\} \tag{3}$$

where Tr denotes the trace operation of a matrix. The value of $L$ is between zero and one. If $L < c_\alpha$, the null hypothesis $H_0$ is rejected, where $c_\alpha$ is the lower 100$\alpha$th percentile of the distribution of $L$. However, under the null hypothesis, the exact distribution of $L$ is unknown. Therefore, some statisticians [18–20] transform $L$ to the statistic $-2\ell nL$, which may be expressed as a chi-square series, as a test criterion by obtaining its asymptotic distribution. Thus, the null hypothesis $H_0$ should be rejected when $-2\ell nL > -2\ell nc_\alpha = UCL$, where $UCL$ is the upper 100$\alpha$th percentile of the distribution of $-2\ell nL$ and is the upper control limit of the multivariate control chart in this paper, which indicates that an assignable cause may exist in the process. To consider the statistical constraints (i.e., the type I error probability and the power) of the chart, the distribution functions of $-2\ell nL$, under the null and alternative hypotheses, should be evaluated.

### 3.1 The Distribution Function of $-2\ell nL$ Under $H_0$

According to Sugiura [20], the distribution function of $-2\ell nL$ under the null hypothesis, which is related to the type I error probability of the multivariate control chart, may be obtained through the following five steps:

Step 1. The characteristic function of $-2\ell nL$ under the null hypothesis is

$$\phi(t) = E\{e^{it(-2\ell nL)}\} = \frac{(2e/n)^{npit}\prod_{g=1}^{p}\left\{\Gamma\left(\frac{n(1-2it)-g}{2}\right)\right\}}{(1-2it)^{np(1-2it)/2}\prod_{g=1}^{p}\left\{\Gamma\left(\frac{(n-g)}{2}\right)\right\}} \tag{4}$$

Step 2. The approximation formula for the gamma function in Eq. (4) is

$$\ell n\Gamma(x+h) = \ell n\sqrt{2\pi} + (x+h-1/2)\,\ell nx - x - \sum_{r=1}^{w}\frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r} + O(x^{-w-1}) \tag{5}$$

where $B_r(h)$ is the Bernoulli polynomial of degree $r$. Taking the logarithmic operation on Eq. (4) and substituting Eq. (5) into Eq. (4) result in

$$\ell n\phi(t) = \left(-\frac{p(p+1)+2p}{4}\right)\ell n(1-2it) - \sum_{r=1}^{w}\frac{(-2)^r B_{r+1}}{r(r+1)n^r}\left[(1-2it)^{-r}-1\right] + O(n^{-w-1}) \tag{6}$$

where

$$B_{r+1} = \sum_{g=1}^{p} B_{r+1}(-g/2).$$

Step 3. Applying the exponential operation on Eq. (6) yields the characteristic function of $-2\ell nL$, i.e., $\phi(t) = e^{\ell n\phi(t)}$, which can be expressed as the summation of a chi-square series [6].

Step 4. According to Theorem 2.6.3 in Anderson [21], if $Z = -2\ell nL$, the density function of $Z$ is

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\phi(t)\mathrm{d}t \tag{7}$$

Step 5. The distribution function of $-2\ell nL$ may be obtained using its definition, i.e.,

$$F(z) = P(-2\ell nL < z) = \int_0^z f(z)\mathrm{d}z \tag{8}$$

As the value of $w$ in Eq. (6) is equal to 3, the distribution function of $-2\ell nL$ under null hypothesis may be expressed as [20]

$$\begin{aligned} F(z) = {} & P(\chi_f^2 < z) + B_2 n^{-1}[P(\chi_{f+2}^2 < z) - \mathrm{P}(\chi_f^2 < z)] \\ & + \frac{1}{6}n^{-2}[(3B_2^2 - 4B_3)P(\chi_{f+4}^2 < z) - 6B_2^2 P(\chi_{f+2}^2 < z) \\ & + (3B_2^2 + 4B_3)P(\chi_f^2 < z)] + \frac{1}{6}n^{-3}[(4B_4 - 4B_2B_3 \\ & + B_2^3)P(\chi_{f+6}^2 < z) + B_2(4B_3 - 3B_2^2)P(\chi_{f+4}^2 < z)] \\ & + B_2(4B_3 + 3B_2^2)P(\chi_{f+2}^2 < z) - (4B_4 + 4B_2B_3 \\ & + B_2^3)P(\chi_f^2 < z)] + O(n^{-4}) \end{aligned} \tag{9}$$

where

$$f = p + \frac{p(p+1)}{2},$$
$$B_2 = \frac{p(2p^2+9p+11)}{24},$$
$$B_3 = \frac{-p(p+1)(p+2)(p+3)}{32} \text{ and}$$
$$B_4 = \frac{p(6p^4+45p^3+110p^2+90p+3)}{480}$$

The distribution function in Eq. (9) is sufficiently accurate only for a large sample size (say, $n \geq 50$). However, in the practical operation of a control chart, the sample size is usually small. Therefore, Eq. (9) cannot be used to obtain the type I error probability for the chart. In this paper, to find the type I error probability for the multivariate control chart, we apply the above-mentioned five steps directly by increasing the value of $w$ in Eq. (6) and run these mathematical operations in the software MATHEMATICA 4.0 [22]. In most cases, as $w \geq 30$, the chi-square series would converge and consequently the type I error probability for the chart can be numerically obtained. The values obtained from MATHEMATICA 4.0 are consistent with those given in Nagarsenker and Pillai [19].

### 3.2 The Distribution Function of $-2\ell nL$ Under $H_1$

In order to evaluate the powers of the multivariate control chart, the distribution function of $-2\ell nL$, under alternative hypotheses, should be studied.

Sugiura [20] developed an asymptotical distribution function of $-2\ell nL$ under $H_1$. However, this function is, again, appropriate only for a large sample size. When this function is expanded, as we do in Section 3.1, it cannot converge to a certain probability value. Therefore, in this paper, we apply simulation and regression approaches to evaluate the distribution function of $-2\ell nL$ under $H_1$.

The simulation and regression procedures for the first out-of-control state (caused by the shift of mean vector) are described as follows:

Step 1. Select a value for *UCL* (called $z$ value) as the critical value of the chart. The selection of $z$ value must meet the statistical requirements. For example, in the case of $n = 4$ and $p = 2$, after the five steps in Section 3.1 are run, we can obtain the upper 10th percentile of the distribution of $-2\ell nL$ under $H_0$ is 14.386. Since the upper bound of the type I error probability of the chart is set 0.1 in this paper, the *UCL* would be greater than or equal to 14.386. When we select a $z$ value, the possible range is $z \geq 14.386$ and, say, 14.40 can be a starting value.

Step 2. Generate $n$ $p \times 1$ random vectors from the multivariate normal distribution with $\mu_1$ and covariance matrix $\Sigma_0$. This step may be done using the software MATHEMATICA 4.0.

Step 3. From the output of Step 1, calculate the value of $-2\ell nL$ using Eqs (1), (2) and (3).

Step 4. Let $m_j = 1$ if the calculated value of $-2\ell nL$ is greater than the selected $z$ value. Otherwise, let $m_j = 0$. The subscript $j$ denotes the run of simulation.

Step 5. Repeat Steps 2–4 by 10 000 times.

Step 6. Compute the simulated power for the selected $z$ value as follows:

$$\text{Power1} = \frac{\sum_{j=1}^{10\,000} m_j}{10\,000} \tag{10}$$

Step 7. Go back to Step 1, select another $z$ value by increasing a step of 0.5 from the last selected $z$ value, and obtain its corresponding simulated power. If the simulated power is greater than 0.9, which is the lower bound of the power for the chart, repeat Step 7; otherwise, stop the simulation procedure.

Step 8. For a set of selected $z$ values and their corresponding simulated powers, treat the $z$ value as an independent variable and the simulated power as the dependent variable, and obtain a polynomial regression equation by using forward selection [23] for a certain combination of $n$ and $p$. This equation is used as a function to estimate the power for the corresponding $UCL$.

For the second out-of-control state caused by the shift of covariance matrix, the same eight steps are applied to estimate and evaluate the power of the chart except that in Step 2, we generate $n$ $p \times 1$ random vectors from the multivariate normal distribution with mean $\mu_0$ and covariance matrix $\Sigma_1$. The simulated power based on the second out-of-control state is denoted by Power2 in this paper.

## 4. The Cost Model

In this section, the cost function given in Montgomery and Klatt [14] and the multivariate loss function developed by Kapur and Cho [16] will be briefly reviewed and will be combined to develop a cost model that is used as the objective function of the design and is intended to be minimised.

According to Montgomery and Klatt [14], the expected total cost per unit of product associated with the test procedure may be written as

$$E(C) = E(C_1) + E(C_2) + E(C_3) \tag{11}$$

where $E(C_1)$ is the expected cost per unit of sampling and carrying out the test procedure, $E(C_2)$ is the expected cost per unit associated with investigating and correcting the process when the test procedure indicates the process is out of control, and $E(C_3)$ is the expected cost per unit associated with producing defective products.

The cost of sampling and testing is assumed to consist of a fixed cost (denoted by $a_1$) independent of the sample size $n$ and a cost per unit sampled (denoted by $a_2$). That is,

$$E(C_1) = (a_1 + na_2)/k \tag{12}$$

where $k$ is the number of units produced between successive samples. To simplify the analysis procedure, we assume that there are only three states for the process, as did Saniga [24]. State 0 denotes the process is in control. State 1 denotes the process mean vector shifts such that the process is out of control. State 2 denotes the process covariance matrix shifts such that the process is out of control. The state in which process mean vector and covariance matrix shift simultaneously, is not considered in this paper. Let $a_3$ be the average cost of investigating and correcting an out-of-control process. Then,

$$E(C_2) = \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)/k \tag{13}$$

where $\rho_i$ is the conditional probability that the test procedure indicates the process is out of control given that the process is in State $i$, for $i = 0$, 1 and 2, and $\alpha_i$ is the probability that the process is in State $i$ at the time the test is performed. Let $a_4$ represent the penalty cost of producing a defective unit of product. Thus,

$$E(C_3) = a_4 \sum_{i=0}^{2} \delta_i \gamma_i \tag{14}$$

where $\delta_i$ is the conditional probability of producing a defective unit given that the process is in State $i$, and $\gamma_i$ is the probability that the process is in State $i$ at any point in time. Therefore, by substituting Eqs (12), (13) and (14) into Eq. (11), the expected total cost per unit of product can be expressed as

$$E(C) = (a_1 + na_2)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)/k + a_4 \sum_{i=0}^{2} \delta_i \gamma_i \tag{15}$$

The $a_i$'s in Eq. (15) are cost coefficients independent of the test procedure. The probability elements $\rho_i$, $\delta_i$, $\gamma_i$ and $\alpha_i$ in Eq. (15) will be examined in the following paragraphs. Further discussion of the general form of the cost function may be found in Montgomery and Klatt [14] and Knappenberger and Grandage [5].

The probability element of $\rho_i$ is the conditional probability that the test procedure indicates the process is out of control given that the process is in State $i$, for $i = 0$, 1 and 2. Note that $\rho_0$ is the type I error probability of the chart, $\rho_1$ is the power due to the shift of process mean vector, and $\rho_2$ is the power due to the shift of process covariance matrix. Therefore, if $UCL = z$, then

$$\rho_0 = 1 - P(-2\ell nL \leq z) \tag{16}$$

where $P(-2\ell nL \leq z)$ may be obtained through the five steps in Section 3.1. Also, $\rho_1$ and $\rho_2$ can be estimated by Power1 (in Eq. (10)) and Power2 respectively using the simulation and regression approaches given in Section 3.2.

The probability element of $\delta_i$ is the conditional probability of producing a defective unit given that the process is in State $i$, for $i = 0$, 1 and 2, whose values depend on the specification limits for each quality characteristic. Define $\mathbf{l}$ and $\mathbf{u}$ as the lower and upper specification-limit vectors respectively, whose elements $\ell_j$ and $u_j$ (for $j = 1, 2, \ldots, p$) represent respectively the lower and upper specification limits of the $j$th characteristic. Therefore, according to the definition, $\delta_i$ may be determined by

$$\delta_0 = 1 - \int_{\ell_1}^{u_1} \dots \int_{\ell_p}^{u_p} \left[ (2\pi)^{p/2} \left| \sum_0 \right|^{1/2} \right]^{-1}$$

$$\exp\left[ -\frac{1}{2}(\mathbf{Y} - \mu_0)^T \sum_0^{-1} (\mathbf{Y} - \mu_0) \right] dy_p \dots dy_1 \quad (17)$$

$$\delta_1 = 1 - \int_{\ell_1}^{u_1} \dots \int_{\ell_p}^{u_p} \left[ (2\pi)^{p/2} \left| \sum_0 \right|^{1/2} \right]^{-1}$$

$$exp\left[ -\frac{1}{2}(\mathbf{Y} - \mu_1)^T \sum_0^{-1} (\mathbf{Y} - \mu_1) \right] dy_p \dots dy_1 \quad (18)$$

$$\delta_2 = 1 - \int_{\ell_1}^{u_1} \dots \int_{\ell_p}^{u_p} \left[ (2\pi)^{p/2} \left| \sum_1 \right|^{1/2} \right]^{-1}$$

$$exp\left[ -\frac{1}{2}(\mathbf{Y} - \mu_0)^T \sum_1^{-1} (\mathbf{Y} - \mu_0) \right] dy_p \dots dy_1 \quad (19)$$

The probability element of $\alpha_i$ is defined as the steady-state probability that the process is in State $i$ at the time the test is performed, for $i = 0$, 1 and 2. To obtain $\alpha_i$, a transition probability matrix $\mathbf{B}$ is required. The elements in $\mathbf{B}$, denoted by $b_{ij}$, are the probability of the process shifting from State $i$ to State $j$ during the production of $k$ units, for $i$, $j = 0$, 1 and 2. Suppose $Q$ units are produced per hour and fractional units can be produced. The probability of remaining in State 0 (in-control state) while $k$ units are produced is $P_0 = \exp(-\lambda k/Q)$. The probability assigned to States 1 and 2 (out-of-control states) while $k$ units are produced is $P_1 + P_2 = 1 - P_0 = 1 - \exp(-\lambda k/Q)$. Knappenberger and Grandage [5] developed a formula to determine the values of $P_1$ and $P_2$ as follows:

$$P_i = \frac{2![1 - \exp(-\lambda k/Q)]\Theta^i(1 - \Theta)^{2-i}}{i!(2 - i)![1 - (1 - \Theta)^2]} \quad \text{for } i = 1 \text{ and } 2 \quad (20)$$

where $0 < \Theta < 1$. Proper selection of the value of $\Theta$ can precisely describe the distribution of $P_1$ and $P_2$. Particularly, $P_1 = P_2$ as $\Theta = 0.667$, $P_1 > P_2$ as $\Theta < 0.667$, and $P_1 < P_2$ as $\Theta > 0.667$. In practice, the values of $P_1$ and $P_2$ may also be determined by past experience. The elements of $\mathbf{B}$ may now be defined. The probability of being in control at the $m$th sample and still in control at the $(m + 1)$th sample is the probability of remaining in control during the production of $k$ units, i.e., $b_{00} = P_0$. The probability of the process being in control at the $m$th sample and being in the $i$th out-of control state at the $(m + 1)$th sample is the probability of shifting to the $i$th out-of control state during the production of $k$ units, i.e., $b_{0i} = P_i$, for $i = 1$ and 2. The probability of the process being in the $i$th out-of-control state (for $i = 1$ and 2) at the $m$th sample and still being in the same out-of-control state at the $(m + 1)$th sample is the probability of detecting the out-of-control state at the $m$th sample times the probability of going the same out-of-control state again for the production of $k$ units plus the probability of not detecting this out-of-control state at the $m$th sample, i.e., $b_{ii} = \rho_i P_i + (1 - \rho_i)$, for $i = 1$ and 2. The probability that the process is in the $i$th out-of-control state (for $i = 1$ and 2) at the $m$th sample and is in control (or in another out-of-control state) at the $(m + 1)$th sample is the probability that the $i$th out-of-control state is detected at the $m$th sample times the probability of remaining in control (or going another out-of-control state) for the production of $k$ units, i.e., $b_{ij} = \rho_i P_j$ for $i = 1$ and 2, $j = 0$, 1 and 2, and $i \neq j$. Therefore, the transition probability matrix $\mathbf{B}$ may be written as

$$\boldsymbol{B} = \begin{bmatrix} P_0 & P_1 & P_2 \\ \rho_1 P_0 & \rho_1 P_1 + (1-\rho_1) & \rho_1 P_2 \\ \rho_2 P_0 & \rho_2 P_1 & \rho_2 P_2 + (1-\rho_2) \end{bmatrix}$$

It is easily shown that $\mathbf{B}$ is the transition matrix of an irreducible aperiodic positive recurrent Markov chain. Therefore, there exists a vector $\boldsymbol{\alpha}$ such that

$$\boldsymbol{\alpha}^T\mathbf{B} = \boldsymbol{\alpha}^T \quad (21)$$

where $\boldsymbol{\alpha}^T = [\alpha_0, \alpha_1, \alpha_2]$, $\alpha_0 + \alpha_1 + \alpha_2 = 1$, and $\alpha_i$ is the steady-state probability that the process is in State $i$ at the time the test is performed, for $i = 0$, 1 and 2. It can be shown that the solution to Eq (21) is

$$\alpha_0 = \rho_1\rho_2 P_0/(\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0) \quad (22)$$

$$\alpha_1 = \rho_2 P_1/(\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0) \quad (23)$$

$$\alpha_2 = \rho_1 P_2/(\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0) \quad (24)$$

The probability element of $\gamma_i$ is defined as the steady-state probability that the process is in State $i$ at any point in time, for $i = 0$, 1 and 2. Duncan [1] has shown that given a shift between the $m$th and $(m + 1)$th samples, the average fraction of time that elapses before the shift occurs is

$$\tau = \frac{1 - (1 + \lambda k/Q)\exp(-\lambda k/Q)}{[1 - \exp(-\lambda k/Q)](\lambda k/Q)}$$

Note that $\tau$ is the conditional expectation of the occurrence of the assignable cause within an interval of sampling. In this paper we assume that during each sampling interval, there exists at most one assignable cause that makes the process out of control. Then, the probability $\gamma_i$ (for $i = 1$ and 2) depends on the probability that the process is in the $i$th out-of-control state when a sample is taken, and the probability that the process is in control when a sample is taken and shifts to this out-of-control state during the production of $k$ units. That is,

$$\gamma_i = \alpha_i + \alpha_0 P_i(1 - \tau) \quad \text{for } i = 1 \text{ and } 2 \quad (25)$$

Consequently, we have

$$\gamma_0 = 1 - \gamma_1 - \gamma_2 = \alpha_0 P_0 + \alpha_0 P_1\tau + \alpha_0 P_2\tau \quad (26)$$

Taguchi and Wu [25] defined product quality as the loss a product imparts to society from the time the product is shipped and, consequently, introduced the quality loss function as a quality performance measure for a product. They indicated that a quadratic loss function sufficiently represents economic loss due to the deviation of quality characteristic from its target. The Taguchi loss function is described mathematically as follows:

$$L(y) = K\,(y - t)^2 \quad (27)$$

where $L(y)$ is the loss associated with the value of quality characteristic $y$, $t$ is the target of the characteristic, and $K$ is a constant depending on the cost at the specification limits and the width of the specification. This loss function recognises the customer's desire to have products that are more consistent, part to part, and a producer's desire to make a low-cost product. Kackar [26] pointed out that the concept of quadratic loss emphasises the importance of continuously reducing process variation. Based on Eq. (27), Kapur and Cho [16] developed a multivariate loss function for the multivariate quality characteristics $y_1, y_2, \ldots, y_p$. The multivariate loss function may be expressed as

$$L(y_1, y_2, \ldots, y_p) = \sum_{i=1}^{p} \sum_{j=1}^{i} K_{ij}\,(y_i - t_i)(y_j - t_j) \tag{28}$$

where $t_j$ is the target of the $j$th characteristic, and $K_{ij}$ is a constant depending on the cost at the specification limits and the width of the specification. Chen [27] gave a complete discussion on determining the values of $K_{ij}$. Specifically, if $y_i$ and $y_j$ are independent, then $K_{ij} = 0$. By applying the operator of expectation on both sides of Eq. (28), the expected loss per unit of product may be obtained as

$$E[L(y_1, \ldots, y_p)] = \sum_{i=1}^{p} K_{ii}[(\mu_i - t_i)^2 + \sigma_i^2] + \sum_{i=2}^{p}\sum_{j=1}^{i=1} K_{ij}[(\mu_i - t_i)(\mu_j - t_j) + \sigma_{ij}] \tag{29}$$

where $\mu_j$ and $\sigma_j^2$ are respectively the mean and variance of $y_j$, and $\sigma_{ij}$ is the covariance of $y_i$ and $y_j$.

In establishing the cost model that is used as the objective function in this paper, we incorporate the multivariate loss function into the cost function in Eq. (15). That is, combining Eqs (15) and (29) results in the average total loss (*ATL*), including both the cost associated with the test procedure and the loss due to deviations of the quality characteristics from their targets, per unit of product as

$$\begin{aligned} ATL &= (a_1 + na_2)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)\Big/ k \\ &\quad + a_4 \sum_{i=0}^{2} \delta_i \gamma_i + a_5\left(1 - \sum_{i=0}^{2} \delta_i \gamma_i\right) \\ &= (a_1 + na_2)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)\Big/ k \\ &\quad + (a_4 - a_5) \sum_{i=0}^{2} \delta_i \gamma_i - a_5 \end{aligned} \tag{30}$$

where $a_5$ is the expected loss per unit of product excluding the penalty cost of producing a defective unit of product; i.e., mathematically, $a_5 = E[L(y_1, \ldots, y_p)] - a_4$. The economic–statistical design of multivariate control charts by considering quality loss determines the three test parameters (i.e., $n$, $k$ and *UCL*) such that *ATL* in Eq. (30) is minimised and the statistical constraints (i.e., $\rho_0 \leq 0.1$, $\rho_1 \geq 0.9$ and $\rho_2 \geq 0.9$) are satisfied.

## 5. An Example and Solution Procedure

From examination of Eq. (30) and the probability elements in the preceding section, it can be seen that determining the optimal three test parameters is not straightforward. To illustrate the nature of the solutions obtained from economic–statistical design of multivariate control charts, a particular numerical example is presented. We assume that only two quality characteristics are of interest (i.e., $p = 2$), that the in-control state is

$$\mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \sum_0 = \begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}$$

and that the other necessary parameters are

$$\mu_1 = \begin{bmatrix} 2\sqrt{10} \\ 2\sqrt{15} \end{bmatrix}, \sum_1 = \begin{bmatrix} 90 & 0 \\ 0 & 135 \end{bmatrix}, \mathbf{l} = \begin{bmatrix} -3\sqrt{10} \\ -3\sqrt{15} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3\sqrt{10} \\ 3\sqrt{15} \end{bmatrix}$$

$\Theta = 0.667$, $\lambda/Q = 0.001$ (i.e., on the average, the process shifts out of control after every 1000 units are produced), $a_1$ = \$20 per sample, $a_2$ = \$0.2 per unit sampled, $a_3$ = \$10 per investigation, $a_4$ = \$10 per defective unit discovered, and $a_5$ = \$5 per non-defective unit.

The solution procedure is in two stages. In the first stage, the feasible solution area of the upper control limit (*UCL*) of the chart for a particular sample size is determined, so that the search area can be narrowed. In the second stage, a grid search is applied to find the values of $n$, $k$ and *UCL* that minimise *ATL*.

### 5.1 The First-Stage Solution Procedure

When the process is in control (i.e., State 0), the quality characteristics **Y** follow a multivariate normal distribution with mean $\mu_0$ and covariance matrix $\Sigma_0$. The type I error probability $\rho_0$ can be obtained through Eq. (16). For a particular sample size ($n$), to meet the statistical constraint of $\rho_0 \leq 0.1$, the feasible solution area for the *UCL* of the chart may be found. Table 1 lists the feasible solution areas of *UCL* for some sample sizes under State 0.

When the process mean vector shifts to $\mu_1$ (i.e., State 1), the quality characteristics **Y** follow a multivariate normal distribution with mean $\mu_1$ and covariance matrix $\Sigma_0$. The power estimation function can be obtained through the simulation and

**Table 1.** The feasible solution areas of the upper control limit for some sample sizes (note that "$z$" indicates a specific value of $-2\ell nL$).

| Sample size (n) | Under State 0 | Under State 1 | Under State 2 | Feasible solution area (intersection) |
|---|---|---|---|---|
| 4 | $z \geq 14.386$ | $z \leq 24.611$ | $z \leq 18.634$ | $14.386 \leq z \leq 18.634$ |
| 5 | $z \geq 12.754$ | $z \leq 30.492$ | $z \leq 24.467$ | $12.754 \leq z \leq 24.467$ |
| 6 | $z \geq 11.914$ | $z \leq 36.682$ | $z \leq 31.076$ | $11.914 \leq z \leq 31.076$ |
| 7 | $z \geq 11.400$ | $z \leq 42.950$ | $z \leq 38.268$ | $11.400 \leq z \leq 38.268$ |
| 8 | $z \geq 11.053$ | $z \leq 49.643$ | $z \leq 46.193$ | $11.053 \leq z \leq 46.193$ |
| 9 | $z \geq 10.802$ | $z \leq 56.367$ | $z \leq 54.164$ | $10.802 \leq z \leq 54.164$ |
| 10 | $z \geq 10.612$ | $z \leq 63.038$ | $z \leq 62.041$ | $10.612 \leq z \leq 62.041$ |

regression approaches mentioned in Section 3.2. For a particular sample size ($n$), to satisfy the statistical constraint of $\rho_1 \geq 0.9$, a power estimation function can be obtained, and then the feasible solution area for the $UCL$ of the chart may be determined. For example, as $n = 6$, the power estimation function is

$$\text{Power1} = 1.02537 - 0.00435849\, z + 0.000221255\, z^2 - 0.0000014173\, z^3 \tag{31}$$

where $z$ is a specific value of $-2\ell nL$. Figure 1 shows the simulated power value and the fitted power estimation function in Eq. (31). The determination coefficient for this fitted function is 0.999426. From this function, the feasible solution area for the $UCL$ of the chart is $z \leq 36.682$. Table 1 lists the feasible solution areas of $UCL$ for some sample sizes under State 1.

When the process covariance matrix shifts to $\Sigma_1$ (i.e., State 2), the quality characteristics $\mathbf{Y}$ follow a multivariate normal distribution with mean $\mu_0$ and covariance matrix $\Sigma_1$. The power estimation function can be obtained through the simulation and regression approaches mentioned in Section 3.2. For a particular sample size ($n$), to meet the statistical constraint of $\rho_2 \geq 0.9$, a power estimation function can be obtained and then the feasible solution area for the $UCL$ of the chart may be determined. For example, as $n = 6$, the power estimation function is

$$\text{Power2} = 0.98825 + 0.00223931\, z - 0.000163437\, z^2 \tag{32}$$

where $z$ is a specific value of $-2\ell nL$. Figure 2 shows the simulated power value and the fitted power estimation function in Eq. (32). The determination coefficient for this fitted function is 0.999222. From this function, the feasible solution area for the $UCL$ of the chart is $z \leq 31.076$. Table 1 lists the feasible solution areas of $UCL$ for some sample sizes under State 2. In Table 1, the last column summarises the feasible solution area of $UCL$ of the chart for a particular sample size by simply examining the intersection of the feasible solution areas under States 0, 1 and 2.

### 5.2 The Second-Stage Solution Procedure

The grid-search approach is used to find the values of $n$, $k$ and $UCL$ that minimise $ATL$. A computer program is coded for this purpose. This program is able to interact with the software MATHEMATICA 4.0 and computes the probability

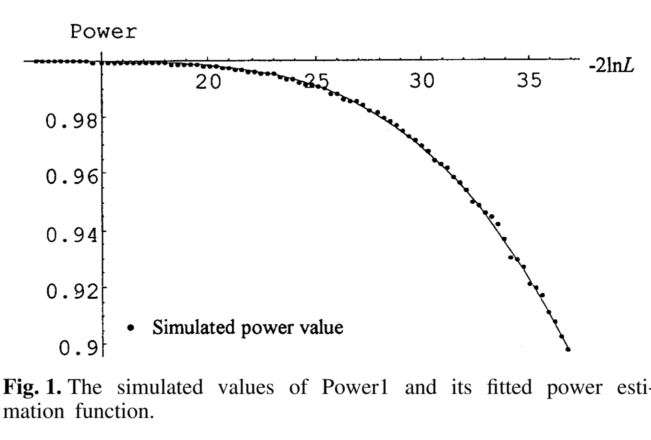


**Fig. 1.** The simulated values of Power1 and its fitted power estimation function.

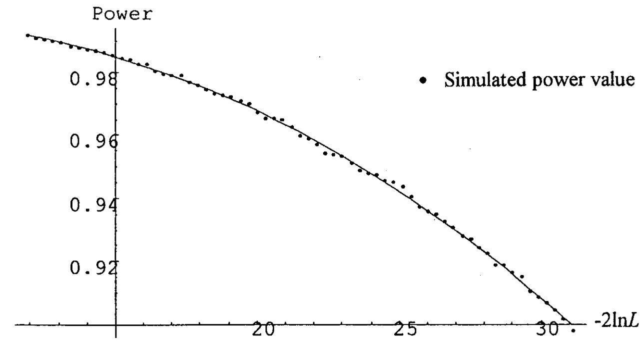


**Fig. 2.** The simulated values of Power2 and its fitted power estimation function.

elements in the loss model. The result from the first-stage solution procedure greatly reduces the range of the search. From the output of the computer program, the optimum solution is $n = 6$, $k = 89$, $UCL = 19.951$, $\rho_0 = 0.009036$, Power1 = 0.998313, Power2 = 0.967871, and $ATL = 5.53167$. That is, the optimal control procedure is to take a random sample of size six every 89 units and conclude that the process is out of control if $-2\ell nL > 19.951$. The expected cost, including the test procedure and quality loss, per unit is \$5.53167.

The two-stage solution procedure of the above example is summarised as follows:

1. In the first stage, the feasible solution space of the upper control limit ($UCL$) for a particular sample size is identified based on the statistical constraints. The lower bound of $UCL$ is determined by the specified type I error probability; while its upper bound is determined by the specified power due to either the shift of mean vector or covariance matrix. Note that the type I error probability for a particular $UCL$ can be obtained using the numerical method described in Section 3.1 and the power for a particular $UCL$ may be estimated using the simulation and regression approach described in Section 3.2.
2. In the second stage, the grid search method is applied to find the appropriate sample size, sampling interval and $UCL$ that minimise the cost.

## 6. Effect of Cost Parameters

In this section, sensitivity analyses are conducted based on the preceding illustrative example to study the effect of cost parameters on the economic–statistical design of the multivariate control charts.

The cost parameter $a_1$ is the fixed cost of taking a sample. Table 2 lists the optimal designs for different values of $a_1$. It can be seen that sampling interval between samples increases as $a_1$ increases. This is consistent with our reasoning.

The cost parameter $a_2$ is the inspection cost per unit. Table 3 lists the optimal designs for different values of $a_2$. As $a_2$ increases, the sample size decreases accordingly. This result is to be expected. In addition, increasing $a_2$ leads to decrease the upper control limit. This tends to stabilise the power with the test.

**Table 2.** Effect of the fixed cost of taking a sample ($a_1$) on the optimal design.

| $a_1$ | $n$ | $k$ | $UCL$ | $ATL$ | $\rho_0$ | Power1 | Power2 |
|---|---|---|---|---|---|---|---|
| 10 | 5 | 63 | 20.182 | 5.40005 | 0.013036 | 0.990311 | 0.935581 |
| 20 | 6 | 89 | 19.951 | 5.53167 | 0.009036 | 0.998313 | 0.967871 |
| 100 | 7 | 215 | 17.409 | 6.06823 | 0.015372 | 0.999948 | 0.989229 |

**Table 3.** Effect of the inspection cost per unit ($a_2$) on the optimal design.

| $a_2$ | $n$ | $k$ | $UCL$ | $ATL$ | $\rho_0$ | Power1 | Power2 |
|---|---|---|---|---|---|---|---|
| 0.1 | 7 | 88 | 21.208 | 5.52477 | 0.004405 | 0.999417 | 0.980753 |
| 0.2 | 6 | 89 | 19.951 | 5.53167 | 0.009036 | 0.998313 | 0.967871 |
| 1.0 | 4 | 96 | 17.070 | 5.57090 | 0.053790 | 0.983794 | 0.915521 |

**Table 4.** Effect of the cost of investigating and correcting the process ($a_3$) on the optimal design.

| $a_3$ | $n$ | $k$ | $UCL$ | $ATL$ | $\rho_0$ | Power1 | Power2 |
|---|---|---|---|---|---|---|---|
| 5 | 5 | 89 | 17.019 | 5.52609 | 0.031650 | 0.996809 | 0.956735 |
| 10 | 6 | 89 | 19.951 | 5.53167 | 0.009036 | 0.998313 | 0.967871 |
| 100 | 7 | 90 | 26.271 | 5.62002 | 0.000796 | 0.997064 | 0.964698 |

**Table 5.** Effect of the quality loss (including $a_4$ and $a_5$) under the condition of $a_4 + a_5 = 15$ on the optimal design.

| ($a_4$, $a_5$) | $n$ | $k$ | $UCL$ | $ATL$ | $\rho_0$ | Power1 | Power2 |
|---|---|---|---|---|---|---|---|
| (8, 7) | 6 | 222 | 20.018 | 3.72854 | 0.008849 | 0.998272 | 0.967584 |
| (10, 5) | 6 | 89 | 19.951 | 5.53167 | 0.009036 | 0.998313 | 0.967871 |
| (12, 3) | 6 | 65 | 19.953 | 7.22567 | 0.009030 | 0.998312 | 0.967863 |

The cost parameter $a_3$ is the cost of investigating and correcting the process. Table 4 lists the optimal designs for different values of $a_3$. As $a_3$ increases, both the sample size and upper control limit tend to increase. This is probably due to the expectation that increasing $a_3$ may correspond to a decrease in type I error probability and an increase in power.

The cost parameter $a_4$ is the penalty cost of producing a defective unit of product and the cost parameter $a_5$ is the expected loss per unit of product excluding the penalty cost of producing a defective unit. Both of them constitute quality loss. Table 5 lists the optimal designs for some combinations of $a_4$ and $a_5$ under the condition of $a_4 + a_5 = 15$. It is noted that as $a_4$ increases, the sampling interval decreases and the expected loss increases, which means that test/sampling should be conducted more frequently such that a high penalty cost can be avoided.

## 7. Conclusions

Although many authors have discussed the economic design of control charts for the last two decades, the design of multivariate control charts still receives relatively little attention in the literature. In this paper, we present the procedure to carry out a economic–statistical design of multivariate control charts by considering quality loss for monitoring the process mean vector and covariance matrix simultaneously. The test statistic $-2\ell nL$ is applied to develop this procedure and the cost model is established based on the cost function given in Montgomery and Klatt [14] and the multivariate quality loss function presented by Kapur and Cho [16]. A numerical example is provided to illustrate the design procedure and the effects of cost parameters on the design are investigated. From the results of the study, we have the following observations.

1. As the fixed cost of taking a sample increases, the sampling interval between samples also increases.
2. As the inspection cost per unit increases, both the sample size and upper control limit lead to decrease.
3. As the cost of investigating and correcting the process increases, both the sample size and upper control limit tend to increase.
4. As the penalty cost of producing a defective unit of product increases, the sampling interval decreases.

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