Atl Econ J (2014) 42:91–107
DOI 10.1007/s11293-013-9400-4

Constrained Mean-Variance Portfolio Optimization with Alternative Return Estimation

Boris Georgiev

Published online: 4 January 2014

Abstract This paper studies the problem of asset allocation in a mean-variance framework. The theoretical model of portfolio optimization is specified and then applied to a long panel data set from historic to most recent times, March 1990 – March 2013. The paper contributes in three ways. First, an alternative asset return model is proposed that combines the historical returns, capital asset pricing model (CAPM) and returns estimated based on firm fundamentals. These return estimates enter the optimization problem. The second contribution is the application of an improved covariance matrix estimator that has superior properties compared to the typical sample covariance estimator. Third, the paper proposes two investments strategies. The first proposition suggests always choosing the maximized Sharpe ratio portfolio and the second one, the portfolio with the highest information ratio. The nature of both strategies is designed for investors with different appetites for risk. The performance of these choices is analyzed in light of four types of constraints: upper/lower investment limits, group constraints and transaction costs. The one-period optimal investment portfolio is rebalanced at quarterly intervals. Both strategies are benchmarked against an alternative investment choice such as holding the S&P 500 index, or investing in a risk-free asset such as a bond. Portfolio analysis and backtesting reveal that the strategies are superior to simply holding an equally weighted portfolio, a risk-free asset or the S&P 500 index.

Keywords Mean-variance optimization · Asset allocation · Investment decision · Finance

JEL G11 · G17

Introduction

Modern portfolio theory began its development almost 50 years ago thanks to the pioneering work of Markowitz (1952). This started a new branch of financial

B. Georgiev (✉)
Department of Economics and Business, School of Business and Social Sciences, Aarhus University, Aarhus, Denmark
e-mail: bageorgiev@gmail.com

economics, dealing with the optimal allocation of financial assets. Every investor decides in which assets to invest depending on his personal risk tolerance and objectives. Some investors (mostly speculators) will accept very high risk for the attractive prospect of greater return. Others (mostly conservative and institutional investors) tolerate lower levels of risk and prefer less volatile portfolios. The "mean-variance" optimization (hereafter referred as MV) searches for the optimal investment allocation, taking into account the trade-off between risk (represented by the variance of returns) and the expected (mean) return of the chosen assets in a portfolio. The optimal portfolio lies on an efficient frontier, which shows the maximum return possible for given levels of risk. Alternatively, the MV optimization can be set to minimize the portfolio variance for a given expected target return.

In this paper I apply the MV framework and seek to optimize the holdings of assets for an investor by also considering the impact of transaction costs and additional portfolio constraints. I predicted expected returns by specifying an alternative model that combines a market model (CAPM), past historical means of returns and analyst predictions based on stock fundamentals. The latter ones are estimated from the fundamental business of each individual firm. For example, such fundamentals are innovation, product lines, market share, and leadership in the company. Thus, this information does not come from a typical historical regression but captures these additional elements of firm performance that are based on the fundamental characteristics of the business as mentioned above. Since in the backtesting, the representative investor rebalances quarterly, it is very important to capture these forward-looking impacts on the firm performance and account for the most recent stock development predictions. The most common practice for return estimation has been to compute the average historical excess return and add it to the risk-free rate. However, solely relying on historical estimates may not be a good choice for a portfolio manager as these estimates are not good predictors of future returns and may produce portfolios with unreasonable weights. Furthermore, I estimated the covariance matrix using a shrinkage covariance estimator (Ledoit & Wolf, 2003) that has desirable properties, such as imposing a low-dimensional factor structure. This is why the shrinkage estimator is a weighted average of the sample covariance matrix with Sharpe's (1963) single-index model estimator where the structure is determined by a shrinkage coefficient k as will be seen in a further section. This approach is different from the traditional case in which the sample covariance matrix is implemented.

I undertook a benchmark investigation where the performance of the optimal portfolio is compared to a naïve strategy (equal weights on all assets), holding a market index strategy (investing in an index) and finally full investment in a risk-free asset. Despite its theoretical appeal, in practice several factors have prevented the theory from gaining wider application. First, the model is very sensitive to even small changes to the inputs and the available estimates may not be accurate enough. Second, in a typical financial institution it is the top management that finally approves portfolio allocation decisions and not quantitative models (Michaud, 1989).

Literature Review

Since the influential contribution of Markowitz, the field has expanded very rapidly and new approaches toward portfolio optimization have been developed. Puelz (2001) performs value-at-risk (VaR) optimization on four model frameworks that apply the VaR approach on ex ante portfolio decisions.

The assumption that historical mean returns could be used as plausible predictors of future expected return has caused wide divergence of opinions in the academic field. Black and Litterman (1992); Merton (1980) and Michaud (1989) show that past returns are poor predictors of future expected returns. Thus reliance on purely historical return estimates may be problematic in the traditional MV optimization framework. These empirical findings stimulated the proposition of alternative ways of expected return estimation. The MV approach established the basis for other influential developments within the field of financial economics such as Sharpe-Lintner's capital asset pricing model (CAPM) (1964; 1965).

Single-index and multi-index models have been used for the return and correlation estimation. These approaches significantly reduce the amount of inputs required for the optimization problem. Dhrymes et al. (1984) report that the number of indices needed is dependent on the number of firms that are included in the analysis. Jorion (1991) compares the accuracy of alternative estimation techniques based on actual data. The paper concludes that the CAPM provides the best expected return estimates.

The view of constant correlation and variance of assets has been challenged by various studies. For example, Flavin and Wickens (2002) use a multivariate GARCH model for the variance-covariance estimation of security returns. Their conclusion is that macroeconomic variables affect the mean-variance inputs across time. Yilmaz (2010) provides evidence for the improved covariance estimation using the DCC-GARCH model for a global minimum variance portfolio by using data from the Istanbul Stock Exchange.

The Model

I start by presenting the model and the allocation problem. The returns of securities are expressed as $R_{i,t+1}:=(P_{i,t+1}-P_{i,t})/P_{i,t}$ for $i=1,\ldots,n$ securities and $P_{i,t}$ is the price of asset i at time t.

Mean-variance analysis holds if either the investor has quadratic utility or the returns of risky assets are normally distributed, $R\tilde{}\mathcal{N}(\mu,\Sigma)$, where μ is a vector of mean returns and Σ is the covariance matrix. The assumption of having normally distributed of returns implies that the expected return is equal to the mean of the historical sample: $E[R_{i,t+1}]=\mu_i$, $i=1,\ldots,n$.

The variance and covariance of returns are specified as:

$$Var\left[R_{i,t+1}\right] = E\left[R_{i,t+1}^2\right]-\left[E\left[R_{i,t+1}\right]\right]^2 = \sigma^2 = \sigma_{ii};\ \ Cov\left[R_{i,t+1},R_{j,t+1}\right] = \Sigma_{ij},\ \ i,j=1,\ldots,n$$

The covariance matrix (Σ) is a $n\times n$ symmetric matrix with the main diagonal being the variance of the individual securities. The covariance of two random variables satisfies the following equality: $|\sigma_{ij}|\leq\sigma_i\sigma_j$, where σ_i is the standard deviation of return of asset i. The same follows for asset j.

Now, let us consider an investor who wishes to allocate optimally a budget amount of $m_t\geq0$, $m_t\in\mathbb{R}$ at time t. All the investments in stocks are non-negative e.g. $x_i \geq 0$, $x_i \in \mathbb{R}$. The investment budget is then equal to $m_t = \sum_{i=1}^{n} x_i P_{i,t}$. Furthermore, the investor requires that all available capital is invested in the appropriate securities so that $\sum_{i=1}^{n} \psi_{i,t} = 1$, where $\psi = (\psi_i, \ldots, \psi_n), \psi_i \geq 0, \psi \in \mathbb{R}$ is a vector of weights representing the percentage share of the total budget invested in security i. Every weight is defined as: $\psi_i = x_i P_{i,t} / m_t$. The value of the current portfolio at time t is $m_t = \Pi_t = \sum_{i=1}^{n} x_i P_{i,t}$. The value of next period's portfolio is represented as: $\Pi_{t+1} = \sum_{i=1}^{n} x_i P_{i,t+1}$. Intuitively the return of the portfolio is expressed as: $R_{\pi,t+1} = (\Pi_{t+1} - \Pi_t)/\Pi_t = \sum_{i=1}^{n} \psi_{i,t} R_{i,t+1}$. Following this, the expected return and variance of the entire portfolio respectively are:

$$E\left[R_{\pi,t+1}\right] = \sum_{i=1}^{n} \psi_{i,t} \mu_i;$$

$$Var\left[R_{\pi,t+1}\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{i,t} \sigma_{ij} \psi_{j,t} = Var\left[\psi_{i,t} R_{i,t+1}\right] + Var\left[\psi_{j,t} R_{j,t+1}\right] + 2Cov\left[R_{i,t+1}, R_{j,t+1}\right] = \sigma_\pi^2;$$

The objective is to minimize the variance of expected returns subject to a lower bound target expected return and the usual constraints:

$$\min_{\pi \in \mathbb{R}^n} Var\left[R_{\pi,t+1}\right] \tag{1}$$
$$\text{s.t} \sum_{i=1}^{n} \psi_{i,t} = 1, \psi_i \geq 0, \quad E\left[R_{\pi,t+1}\right] = \sum_{i=1}^{n} \psi_i \overline{R}_i \geq l_2$$

where l_2 is the lowest acceptable expected return. This problem has a quadratic objective function and linear constraints. Equation (1) can be solved using the Lagrange method or by using quadratic programming with the algorithm suggested by Goldfarb and Idnani (1983). The optimal weights of each asset can be obtained by solving the problem in Eq. (2) using the Lagrange multiplier method. The Lagrangian is provided for the case of n assets:

$$\min_\psi \left(\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n} \psi_{i,t} \sigma_{ij} \psi_{j,t} - \lambda \left(\sum_{i=1}^{n} \psi_{i,t} - 1 \right) - \phi \left(\sum_{i=1}^{n} \psi_i \overline{R}_i - l_2 \right) \right) \tag{2}$$

where l_2 is the minimum required return, $\overline{R}_i$ is the expected return of asset i, λ and ϕ are Lagrange multipliers. The reason why one half is added in front of the portfolio variance expression is that it makes the solution more tractable and easier to work with. By forming the partial derivatives with respect to every asset weight ψ_i, $n+2$ equations have to be solved for $n+2$ unknowns: $\psi_{i=1,\ldots,n}$, λ and ϕ. Thus the solution to the Markowitz problem involves solving the system of linear equations and solving for every individual weight and the Lagrange multipliers. I used numerical methods in Matlab to find the optimal combination of assets and weights.

Investors need plausible predictions of the future development of stock prices, their correlations and variances. Stock returns are assumed to be random variables, varying over time. Then how can the terminal value of the investment be maximized if it is stochastic? In order to partially resolve this problem I relied on the expected value of investments. One formulation of the problem could be to maximize the difference between the current investment amount m and the future expected value of the portfolio. By doing so, the investor only focuses on the return but neglects the associated risk. Uncertainty has to be factored in the optimization process because the expected value may not be hit exactly, meaning that the actual difference between the current and terminal investment value may be smaller or greater than expected by the investor. This “error” may occur due to the dispersion of all the possible future returns in our portfolio. The key task is to see whether it is possible to “hit” the expected value with greater certainty e.g. reduce the dispersion of possible outcomes. The essence of MV optimization theory is to find the trade-off between risk and return that is optimal for an invetor.

Data

This paper uses in total 5,796 daily observations on 30 different traded securities and the S&P 500 index for the period 4 March 1990 to 5 March 2013. This index is also used as a benchmark, since it includes mostly large cap stocks which are comparable to the 30 equities that I consider in the optimization. The stocks have been chosen based on liquidity and trading activity. The daily closing prices are adjusted for splits and dividends. Summary statistics and the list of equities are available in Table 4.

A quarter of the included large-cap companies were publicly offered after 1990. This means that eight of the considered companies have missing data. Missing observations for some variables may cause serious problems when estimating the expected returns and variances and when backtesting the strategies. I dealt with this problem by using the ECM (expected conditional maximization) algorithm with the maximum likelihood estimation (Xiao-Li & Rubin, 1993). All the considered stocks are also grouped into specific sectors. This classification enabled me to implement additional group constraints. The industrial groups are: *technology, aviation & defense, consumer goods, financial services, telecommunications, energy and pharmaceuticals*.

Throughout the period the hard impact of the Dot-com bubble from the 2000s and the global financial crisis from 2008 has had a very strong impact on the equity prices, leading to a big decline in stock prices and indices.

Estimation of Returns and Covariances

Estimation of the expected returns and variance of assets are focal ingredients in the optimization model. Variance estimation based on historical values yields tolerable results that are close to reality. Nevertheless, proper return estimation based on historical samples is not accurate and is almost impossible due to estimation error. For that reason, I contributed with an alternative estimation model. The approach takes a combination of the CAPM prediction, historical means and investor’s personal confidence in the future price movement of a particular stock.

Since its introduction, the CAPM has been both highly admired and criticized. On one hand, its fame stems from the easy to understand predictions about asset returns and also the simplifying assumptions. In the presence of a risk-free asset, the optimal set becomes a straight line originating from the risk-free rate that intersects the efficient frontier. The optimal portfolio can be seen as a combination of a riskless asset and risky portfolio that is tangent to the efficient set. Such an efficient portfolio can also be constructed by combining a risk-free asset and a mutual fund. The Sharpe-Lintner CAPM takes the following form:

$$E[R_i] = R_f + \beta_{i,m}\left(E[R_m]-R_f\right), \;\; i = 1, \ldots, n; \;\; \beta_{i,m} = \frac{cov(R_i, R_m)}{\sigma^2(R_m)}$$

Assets that are uncorrelated with the market ($\beta_{i,m}$=0) have an expected return of the risk-free rate, R_f. For those that are correlated with the market, the risk premium is determined by the market β. It measures the sensitivity of the asset return in relation to the market. The expected return of a market portfolio has a β=1. The predictions of the model are straightforward and the main insight is that the return of an individual asset is only a function of the market and R_f. Other models have been developed to account for additional factors that have an influence on the expected return of assets (see "Three-factor" model by Fama and French (1996)).

In order to test the predictions of the CAPM, the model can be estimated either by cross-section or time series regressions. I estimated the time-series version of the model. This version of the CAPM includes an intercept α introduced by Jensen (1968). The econometric model takes the following form:

$$R_{it}-R_{ft} = \alpha_i + \beta_{i,m}\left(R_{mt}-R_{ft}\right) + \varepsilon_{it}, \;\; i = 1, \ldots, n; \;\; t = 1, \ldots, k \tag{3}$$

where ε_{it} is random noise with $E[\varepsilon_i]$=0, $E[\varepsilon_i\varepsilon_j]$=$\sigma_{ij}$, i,j=1,…,n and σ_{ij} is called non-systematic variance (residuals). A brief glance at the original CAPM equation shows that α should not be significantly different from zero. Any non-zero values are assumed to be due to disequilibrium in the market which does not last for long. A lot of current portfolio management is devoted to looking for exploitable positive alpha assets on the market. If an asset has a positive alpha, this signals that there is an available excess return, unrelated to the market. I estimated the parameters using Maximum-Likelihood (ML). The estimation window is from 5 March 2007 until 4 March 2013. For this period there is no missing data for any of the stock returns. However, when performing the backtesting of the considered strategies in the penultimate section, some of the stocks were not available in the beginning of 1990. Thus when setting the rolling efficient frontier some of the assets are excluded from the portfolio analysis since they were not on the stock exchange. I used the ECM algorithm with the ML estimation to deal with the missing data issue for some stocks. The approach does not discard stocks with incomplete data but simply tries to estimate the moments conditional on the available data. Most of the missing data is from the financial industry where most of firms became public in the late nineties. The multivariate normal linear regression model takes the form: $Z \sim \mathcal{N}(X_m w, C)$, where Z is a vector of independent random variables, m=1,…,n observations, is a X_m design matrix including the explanatory variables, C is the covariance matrix and w is a p dimension vector of parameters. Furthermore the regression residuals are assumed to be normally distributed with zero mean and a $n \times n$ covariance matrix C for each observation from m. The parameter values for C and w are obtained by maximizing the log-likelihood function of the model.

The estimated model in Eq. (3) is run separately for all 30 equities. The adjusted β in Table 1 is calculated as: $Adj.\beta{=}0.67\times\beta{+}0.33\times1$. The adjusted β captures the effect of mean reversion towards the market average where $\beta{=}1$. In this paper I took the average annualized yield rate of the 5 year Treasury bill as the risk-free rate. For the considered 6 year estimation period, $R_f{=}2.16\%$. To obtain the expected market return, I used the Thomson-Reuters estimate of 7.6 % year-to-date growth prospect for 2013. This value of the expected annual market return enters the CAPM estimation. For the fundamentally predicted returns, R^{ef}, I used the company reports and analyst predictions for the considered equities throughout the period.

Table 1 β and R^e estimation (2007–2013) based on $R_{m,t+1}{=}7.6\%$ and $R_f{=}2.16\%$

	α		β		Adj. β	σ_ε		R^e	R_C	R_H
Equity	Coeff.	T-stat.	Coeff.	T-stat.	Coeff.	Coeff.	Std. error	Value	Value	Value
IBM	0.0005	−2.0437	0.739	−44.9037	**0.818**	0.01	−0.0019	0.11	0.067	0.153
AAPL	0.001	−2.137	0.959	−32.1954	**0.963**	0.0182	−0.0035	0.214	0.075	0.271
MSFT	0	−0.0951	0.913	−42.8472	**0.933**	0.013	−0.0025	0.081	0.072	0.027
TXN	0.0001	−0.2123	0.9	−36.7672	**0.924**	0.0149	−0.0028	0.061	0.072	0.04
DELL	−0.0004	−0.7263	0.973	−29.4197	**0.972**	0.0202	−0.0038	0.015	0.074	−0.076
INTC	0.0001	−0.3573	1.019	−45.54	**1.003**	0.0136	−0.0026	0.092	0.076	0.051
GOOG	0.0003	−0.8747	0.899	−35.3563	**0.923**	0.0155	−0.003	0.087	0.072	0.107
AMZN	0.0013	−2.056	1.084	−27.9053	**1.045**	0.0237	−0.0045	0.173	0.08	0.335
HPQ	−0.0005	−1.0223	0.934	−33.0412	**0.946**	0.0172	−0.0033	0.004	0.073	−0.095
ORCL	0.0005	−1.3172	0.995	−44.6718	**0.987**	0.0136	−0.0026	0.093	0.076	0.135
SAP	0.0004	−1.0314	0.963	−41.2163	**0.966**	0.0143	−0.0027	0.097	0.075	0.114
LUV	−0.0002	−0.4153	0.918	−28.1618	**0.936**	0.0199	−0.0038	0.038	0.072	−0.034
BA	0	−0.1177	0.973	−42.3386	**0.972**	0.014	−0.0027	0.06	0.074	0.008
LMT	0	−0.0288	0.674	−31.8144	**0.775**	0.0129	−0.0025	0.041	0.064	0.017
NOC	0	−0.1328	0.751	−36.8962	**0.826**	0.0124	−0.0024	0.067	0.067	0.03
GE	−0.0002	−0.4422	1.168	−46.3676	**1.101**	0.0154	−0.0029	0.05	0.081	−0.026
PG	0.0002	−0.7563	0.552	−38.4217	**0.694**	0.0088	−0.0017	0.046	0.06	0.063
UL	0.0004	−1.1328	0.701	−33.7365	**0.793**	0.0127	−0.0024	0.089	0.065	0.113
KO	0.0004	−1.4426	0.552	−33.2109	**0.694**	0.0101	−0.0019	0.09	0.06	0.115
PEP	0.0002	−0.6907	0.509	−32.1713	**0.666**	0.0096	−0.0018	0.061	0.058	0.063
GS	−0.0002	−0.3218	1.442	−42.7631	**1.282**	0.0206	−0.0039	0.082	0.091	−0.025
UBS	−0.0009	−1.459	1.783	−45.2621	**1.507**	0.024	−0.0046	−0.057	0.103	−0.21
DB	−0.0007	−1.1797	1.886	−51.9078	**1.575**	0.0222	−0.0042	0.043	0.107	−0.153
BLK	0.0003	−0.7114	1.392	−46.263	**1.249**	0.0184	−0.0035	0.105	0.09	0.103
T	0.0002	−0.5592	0.778	−44.9537	**0.843**	0.0106	−0.002	0.03	0.068	0.058
VZ	0.0003	−1.1085	0.707	−38.8463	**0.797**	0.0111	−0.0021	0.061	0.065	0.1
XOM	0.0002	−0.6618	0.95	−55.4329	**0.957**	0.0104	−0.002	0.081	0.074	0.064
ACT	0.0007	−2.0088	0.627	−27.2171	**0.744**	0.014	−0.0027	0.185	0.063	0.203
GSK	0	−0.0108	0.623	−31.5916	**0.741**	0.012	−0.0023	0.096	0.062	0.019
NVO	0.001	−2.219	0.681	−24.6917	**0.779**	0.0168	−0.0032	0.201	0.065	0.262

In Table 1 most of the α estimates are near zero as expected, even though some of the stocks display α values that are significantly different from zero. All t-statistics are highly significant at the 1 % level. As expected the financial services firms are by far the riskiest with $\beta>1$. This means that in good times these stocks shall outperform the market and in bad times experience greater drawdowns than the market. Column five of Table 1 includes the standard deviation of the residual. This is an estimate of non-systematic risk.

This section of the paper shows how estimates could be obtained for a one-period decision. This is done p times depending on how often the investor wants to rebalance. My approach involves a novel combination of several inputs for the final expected return estimation. I devised a methodology where the expected return for a given stock, R_i^e, is determined by the weighted average of the historical average return, the expected return obtained from the CAPM and finally a confidence weighted prediction based on fundamentally predicted return of the equity. The model is:

$$R_i^e = \frac{R_{H,i}\left[\frac{|R_{H,i}|}{|R_{H,i}|+|R_{C,i}|}\right]+R_{C,i}\left[\frac{|R_{C,i}|}{|R_{H,i}|+|R_{C,i}|}\right]}{1+\gamma}+\frac{R_i^{ef}\gamma}{2} \tag{4}$$

where $R_{H,i}$ is the average historical return for a given range of past values, $R_{C,i}$ is the CAPM estimated return, R_i^{ef} is the expected return based on fundamentals, $\gamma=0,\ldots,1$ is a variable that indicates confidence/agreement with the fundamentally predicted return, R_i^e is the estimated return that enters the MV optimization and the terms in the square brackets are the relative weights. The absolute values are taken so that the weights have meaningful signs and are always positive. Estimates for the fundamentally predicted return could come from various information sources, analyst reports, etc. When $\gamma=0$ (disagreement with the analyst outlook), Eq. (4) reduces to $R_{H,i}\left[\frac{|R_{H,i}|}{|R_{H,i}|+|R_{C,i}|}\right]+R_{C,i}\left[\frac{|R_{C,i}|}{|R_{H,i}|+|R_{C,i}|}\right]$. However if $\gamma\neq0$, then R_i^{ef} gets weight in the final expected return value that is to be used in the mean-variance optimization. As the investor becomes more confident (γ increases) about the fundamentally predicted return (R_i^{ef}), the more R_i^e approaches R_i^{ef}. In the case where $\gamma=1$, the equation becomes an average of the two right-hand side fractions in Eq. (4). This property is formidable as it never ignores the historical average and the CAPM prediction but could potentially discard R_i^{ef} if such estimates are not available or the investor does not share the same view on future stock returns.

Portfolio Analysis and Backtesting

I initially assumed that the representative investor holds an equally-weighted portfolio, consisting of all 30 assets. This means that the weight in every asset is $\psi_i=0.03\overline{3}$ for $i=1,\ldots,30$. The task is to find a more profitable strategy than the equal weight portfolio. For example, an investor could hold an index such as the S&P 500. These options are tested and compared to the proposed strategies. The first one is to always hold the portfolio that maximizes the Sharpe ratio

(Sharpe, 1964). This is used when the purpose is to maximize the gross portfolio return. Alternatively, the objective could be to maximize the excess return over a benchmark index or portfolio, which is the *Max I* proposition. Both of these performance metrics are widely used and thus provide a good benchmark statistic (Bacon, 2012). Both are defined as:

$$S_\pi = \frac{E\left[R_{\pi,t} - R_{f,t}\right]}{\sqrt{Var\left(R_{\pi,t} - R_{f,t}\right)}} \quad I_\pi = \frac{E\left[R_{\pi,t} - R_{b,t}\right]}{\sqrt{Var\left(R_{\pi,t} - R_{b,t}\right)}}$$

where R_π is the portfolio return, R_f is the risk-free rate, the nominator is the expected excess return and R_b is the return of a selected benchmark portfolio. Both criterions are similar but S measures the excess return over R_f, whereas I takes the excess return over a benchmark which in this case is the market (S&P 500). The nominator is also called "active return" and the standard deviation of the active return - "tracking error". A potential caveat is the hard comparison between two ratios. For example there is no indication how much better is S= 1.2 vs. S=0.8.

The strategy considered in this paper is that the investor re-optimizes the portfolio quarterly and estimates the expected returns and covariance matrix repetitively. Having these inputs, the next step is to choose the optimal portfolio according to a criterion. I incorporated several constraints which limit excessive investment exposure in specific groups of assets:

1. *Upper bound group constraints*: Up to 60 % of the investment budget can be invested in Technology (G_{tech}) stocks; up to 20 % can be invested in Aviation & Defense (G_{ad}) equity and up to 15 % in stocks of Financial services (G_{fs}) firms.
2. *Lower bound group constraints*: At least 10 % of the investment budget is to be invested in each of the following groups: Technology (G_{tech}), Aviation & Defense (G_{ad}) and Financial services (G_{fs}).
3. *Limits*: Up to 30 % of the entire budget can be invested in a single equity and no negative weights are allowed.
4. *Transaction costs* are modeled linearly as a function of the current equity price, $\tau = f(P_{i,t})$. Buying and selling costs are assumed to be the same and are set to 12 basis points. These are often seen costs that are equal to the commission over the stock price that the investor buys/sells.

G_{tech}=(Asset$_1$,…,Asset$_{11}$), G_{ad}=(Asset$_{12}$,…,Asset$_{16}$) and G_{fs}=(Asset$_{21}$,…,Asset$_{24}$) are the respective groups that include assets from the same industry. The assets can be seen in Table 4. The mathematical formulation of the investment rules are: $G_{tech} \cdot v' \leq 0.5$; $G_{ad} \cdot b' \leq 0.3$ and $G_{fs} \cdot v' \leq 0.2$, where $v = \{\psi_1, \ldots, \psi_{11}\}$, $b = \{\psi_{12}, \ldots, \psi_{16}\}$ and $c = \{\psi_{21}, \ldots, \psi_{24}\}$ are vectors with the respective weights for each group. The limits constraint is $0 \leq \psi_{1,\ldots,30} \leq 0.30$. Transaction costs are an important ingredient that can heavily influence the asset allocation solution. A possible extension of my framework could be to incorporate non-linear transaction costs.

For the estimation of the covariance matrix of stock returns, I adopted an improved covariance matrix estimator. This paper builds upon the framework of Ledoit and Wolf (2003) and uses their improved shrinkage estimator. Their estimate is a weighted average of two alternative covariance estimators: the sample covariance matrix and a single-index matrix[1]. This approach is known as shrinking. The general form of the shrinkage covariance estimator is $\widehat{S}=\frac{k}{T}I+\left(1-\frac{k}{T}\right)S$, where $\widehat{S}$, the improved covariance matrix, is applied in the paper, S is the sample covariance estimate, I is the covariance matrix derived from a single-index model of stock returns, and $\frac{k}{T}\in[0;1]$ is the shrinkage intensity (see Ledoit and Wolf (2003)). I took the square root of the diagonal entries, σ_{ii} from the covariance matrix as the estimated standard deviations of the returns. The shrinkage covariance estimator in this paper has shown to always produce an invertible and well-specified covariance matrix, which is of big practical importance. The only component that is crucial in practice is the estimation of the shrinkage intensity. Ledoit and Wolf (2003) obtain an estimate of around 0.75, which is also the value I used. This would indicate that there is roughly three times more estimation error in the sample covariance matrix than bias in the single-index model covariance matrix. The proposed estimation approach also permits to account for extra market covariance without the need of introducing a new factor into an index model.

Constrained Optimization

Next, I studied the impact of the four constraint rules on the asset allocation problem in the MV framework. In Fig. 1 the efficient frontier with a risk-free asset does not extend until the vertical axis. The reason is that the investor imposes the limit constraint that at least 10 % shall be invested in each of the three groups of stocks. The minimum risk portfolio on the dotted efficient frontier includes 70 % investment in a risk-free asset and 30 % invested in risky assets. The "upper/lower bound group" constraint is binding.

The efficient frontier on the left side of Fig. 1 incorporates transaction costs. The global minimum variance portfolio in the presence of a risk-free asset has a return, R_π= 0.55% and standard deviation, σ_π=7.83%. When an investment in a riskless asset is not possible then the lowest possible standard deviation that could be attained is σ_π= 18.01% and R_π=3.50% as can be seen in the left panel of Fig. 1.

I proceeded with the analysis of the proposed strategies when transaction costs are considered. The Sharpe portfolio offers a return of 14.45 % and risk of 23.34 %, indicated by the point on the left panel of Fig. 1. By imposing the constraints, the expected return falls by 6.99 % points and risk decreases by only 0.02 % compared to the right side of Fig. 1.

Table 2 outlines the initial equal-weighted portfolio and presents the characteristics of the maximized portfolios. The initial portfolio is dominated by *Max S*. First, the maximized Sharpe portfolio offers a much greater return for a lower level of volatility. Second, the *Max I* portfolio has the lowest risk compared to the equal weights and *Max S* and offers almost twice as small volatility. Risk-averse investors would find the strategy of following the *Max I* portfolio more attractive.

[1] Their shrinkage estimator aims at attaining an optimal trade-off between an unbiased and a biased estimator. One way this could be done is by appropriately weighing (shrinking) both estimators. This idea is in the base of Ledoit and Wolf (2003). This procedure is better known as shrinking the unbiased estimator towards a fixed target that is in fact represented by the biased estimator.

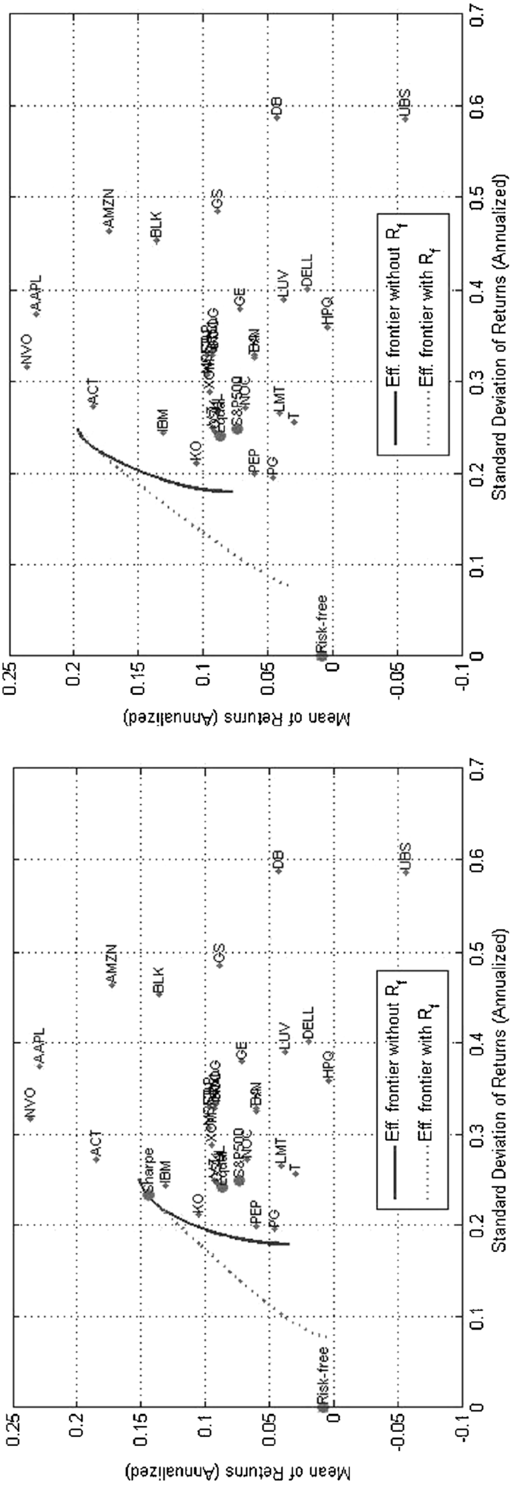

Fig. 1 Efficient frontiers With (*Left*) and without (*Right*) transaction costs

Table 2 Portfolio characteristics for different targets with all constraints

Portfolio strategy	*Max S*	*Max I*	$\psi_i = 0.03\overline{3}$
Return, R_π	14.45 %	6.67 %	8.69 %
Std. deviation, σ_π	23.34 %	11.23 %	24.13 %

In contrast to the case, where no constraints are implemented the Sharpe ratio fell from almost 0.9 down to around 0.6 after imposing the constraints. The same can be said for the information ratio portfolio. The maximum is around one versus 0.6. This hit is primarily due to the lower and upper bounds on stock holdings and the transaction costs. These rules limit the weights distribution but do not alter significantly the risk profile. In comparison to the equal weight portfolio there is room for improvement. Even with investment constraints the proposed strategies are superior.

Strategy Backtesting

This last section of the paper presents the historical performance of the strategies when applied continuously through the sample period from March 1990 to March 2013. It is important to note that in order to implement the novel return estimation model in Eq. (4) the investor has to specify R_i^{ef} and the confidence level γ for each security at every re-balancing point. Not all estimates are available because not all equities were listed on the stock exchanges since the beginning. For that reason, the algorithm that I applied uses 504 days (equivalent to two trading years) of historical stock data for the estimation of expected returns in the rolling window. Each efficient frontier is formed by estimating 300 optimal portfolios. In 1992 the allocation problem combined only 23 equities since seven from the asset universe were not available. By 2013 all the assets enter the optimization. In Fig. 2 the performance of the Sharpe strategy outperforms the market in cumulative terms and only displays slightly suboptimal performance around 2001.

Figure 2 shows that before the dot-com bubble burst, efficient portfolios at the high end of the curve offered very high return for a given level of risk as displayed by the upper area of the surface. The cumulative value of a dollar invested in the beginning of the period is shown on the right. The financial performance of the market and the chosen portfolio has not been smooth but nevertheless significantly outweighs the risk-free holdings yield. By investing a single dollar in a portfolio that follows the *Max S* strategy, an investor would have earned today around eight dollars. The efficient frontier varies considerably in shape across time, so when using the Max *S* portfolio the investor would have to heavily rebalance and be exposed to different levels of risk. The Sharpe ratio is a more aggressive strategy that would appeal more to risk-lovers.

Finally, I assess the performance of the information ratio portfolio. The main difference between *S* and *I* is that the optimal region is much more stable in the latter case. Figure 3 displays the "calm" dark region in the lower part of the efficient frontier. It is evident that the *Max I* portfolio has the same risk level

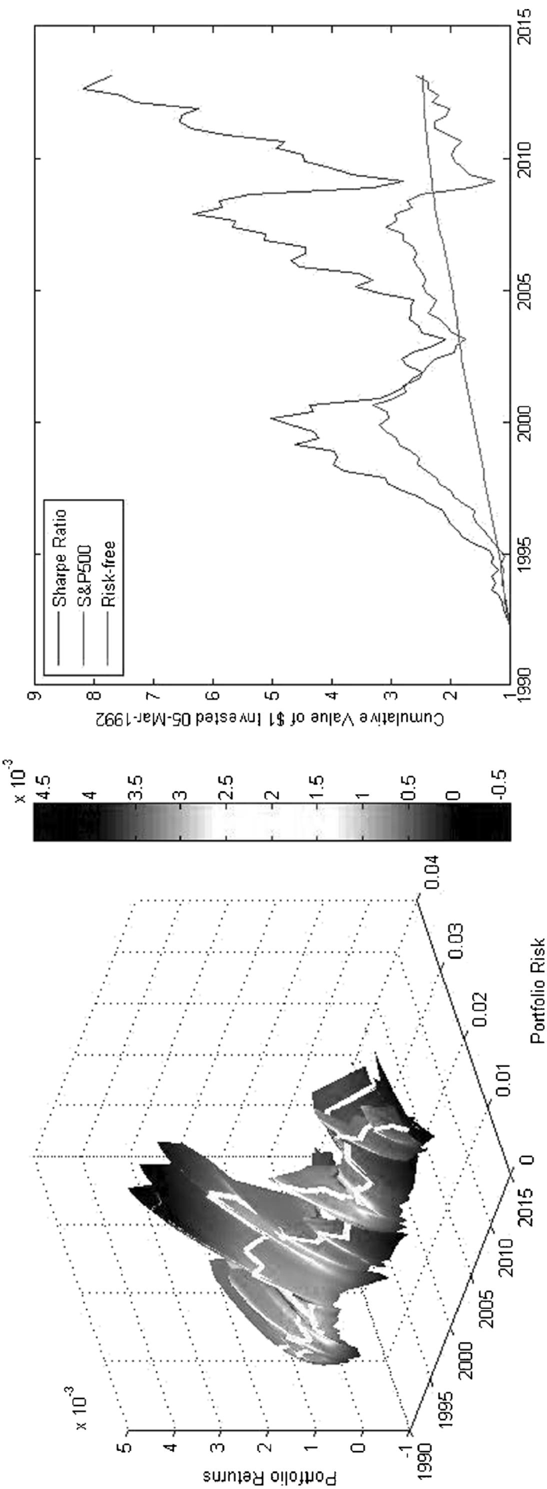

Fig. 2 Evolution of efficient frontier, sharpe portfolio and strategy performance

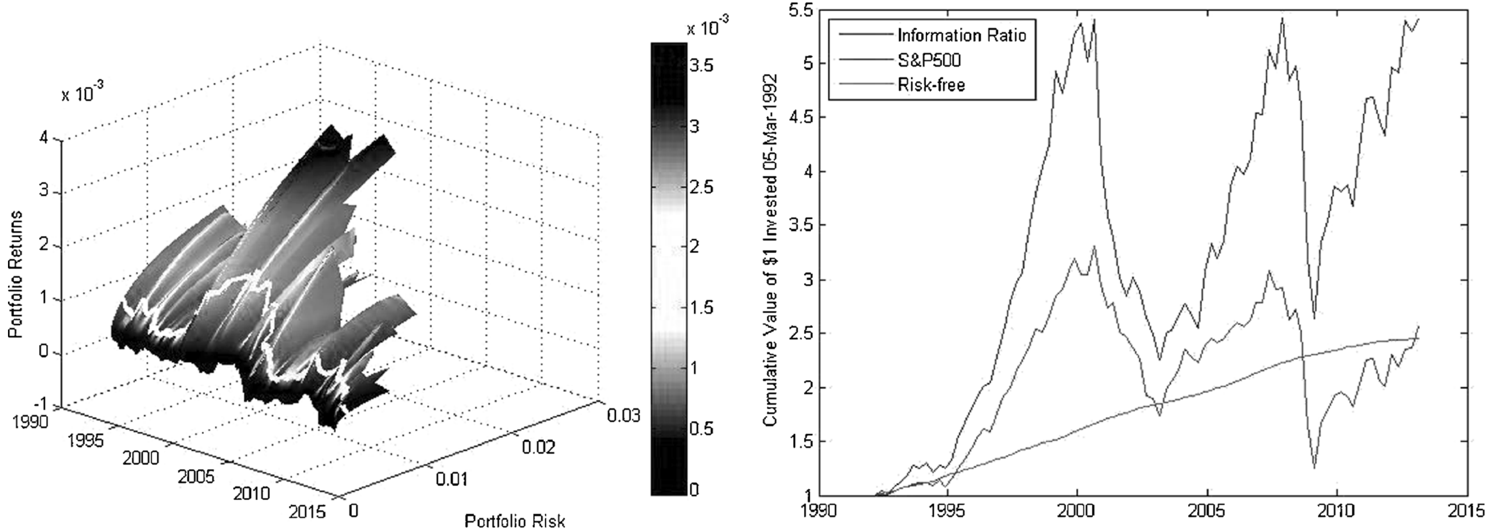

Fig. 3 Evolution of efficient frontier, informaton portfolio and strategy performance

Table 3 Performance of strategies

Investment	Average return	Std. deviation	Geometric return	Max. Drawdown
Max *S*	12.26 %	21.75 %	10.21 %	58.54 %
Max *I*	9.83 %	18.37 %	8.36 %	58.40 %
R_f	4.30 %	0.92 %	4.37 %	0.00 %
S&P 500	5.99 %	16.89 %	4.58 %	62.17 %

across time and the returns follow closely the market performance. If the return of the portfolio is zero it means that it is equivalent to holding the market. By investing a single dollar in the beginning of the period an investor could have made 5.4 dollars.

Table 3 reports the arithmetic mean and standard deviation of returns per year in columns one and two. The final column indicates the maximum drawdown of the strategy – the maximum value loss that has been incurred over the entire period. The offered strategies have shown formidable results and even by incorporating constraints, the performance is superior to the market and the risk-free investment. The Sharpe ratio strategy entails greater risk by having higher standard deviation than the Info ratio strategy, the market and the risk-free asset. The *Max S* portfolio dominates the other investment options by also offering the highest average expected return. Tables 5 and 6 contain the exact weights on the selected equities in constrained and unconstrained portfolios for both *Max S* and *Max I* portfolios.

Conclusion

This research paper considers the problem of optimal asset allocation faced by a representative investor. The key question is whether an investor can do better than simply holding an equal weight or a market portfolio. I provided an affirmative answer to this question by proposing two separate strategies. These are the maximized Sharpe and information ratio portfolios.

In portfolio analysis one of the main problems is to obtain sound predictions of expected future returns and covariance among the assets. I contributed by proposing an alternative model for the estimation of expected returns. The approach takes weighted averages of the historical return, the CAPM predicted return and investor specified return outlook. I estimated the covariance matrix by using an improved shrinkage estimator that performs better than the sample covariance matrix. I incorporated four sets of constraints and study their impact on the final performance of the strategies. As a result, both suggested portfolios (Maximized Sharpe and information ratio) yield positive results and provide evidence for superior returns, compared to other alternative investment options such as holding a market index or investing in a risk-free asset. The final choice between the two strategies depends on the risk preferences of the investor.

Acknowledgments I thank Professor Bent Jesper Christensen for his advice, comments and useful suggestions during the writing of this paper. As well I extend my gratitude to Post Doc Alexey Rubtsov for the fruitful discussions on various financial topics and useful suggestions for improvement and to Professor Gary Clayton for the additional comments and advice for further amelioration of the paper.

Appendix

Table 4 List of equities, Yearly historical average returns, std. deviations and data range

	Ticker	Stock	Classification	Mean ret.	Std. dev.	Data range
1	IBM	IBM Corp.	Technology	0.107	0.291	2013–1990
2	AAPL	Apple Inc.	Technology	0.174	0.494	2013–1990
3	MSFT	Microsoft Inc.	Technology	0.171	0.338	2013–1990
4	TXN	Texas Instruments Inc.	Technology	0.136	0.445	2013–1990
5	DELL	Dell Inc.	Technology	0.237	0.509	2013–1990
6	INTC	Intel Corp.	Technology	0.135	0.408	2013–1990
7	GOOG	Google Inc.	Technology	0.249	0.342	2013–2004
8	AMZN	Amazon.com Inc.	Technology	0.322	0.674	2013–1997
9	HPQ	Hewlett-Packard Co.	Technology	0.081	0.396	2013–1990
10	ORCL	Oracle Corp.	Technology	0.18	0.52	2013–1990
11	SAP	SAP AG	Technology	0.032	0.457	2013–1998
12	LUV	Southwest Airlines Co.	Aviation & Defense	0.108	0.384	2013–1990
13	BA	Boeing Co.	Aviation & Defense	0.075	0.312	2013–1990
14	LMT	Lockheed Martin Corp.	Aviation & Defense	0.116	0.276	2013–1995
15	NOC	Northrop Grumman Corp.	Aviation & Defense	0.143	0.282	2013–1990
16	GE	General Electric Co.	Aviation & Defense	0.094	0.294	2013–1990
17	PG	Procter & Gamble Co.	Consumer goods	0.121	0.239	2013–1990
18	UL	Unilever PLC	Consumer goods	0.115	0.245	2013–1990
19	KO	The Coca-Cola Co.	Consumer goods	0.114	0.238	2013–1990
20	PEP	Pepsico Inc.	Consumer goods	0.114	0.249	2013–1990
21	GS	The Goldman Sachs Group Inc.	Financial services	0.066	0.42	2013–1997
22	UBS	UBS AG	Financial services	−0.03	0.445	2013–1998
23	DB	Deutsche Bank AG	Financial services	0.007	0.453	2013–1998
24	BLK	BlackRock Inc.	Financial services	0.228	0.391	2013–1999
25	T	AT&T Inc.	Telecommunications	0.083	0.271	2013–1990
26	VZ	Verizon Communications Inc.	Telecommunications	0.078	0.263	2013–1990
27	XOM	Exxon Mobil Corp.	Energy	0.118	0.241	2013–1990
28	ACT	Actavis Inc.	Pharmaceuticals	0.125	0.401	2013–1993
29	GSK	GlaxoSmithKline PLC	Pharmaceuticals	0.097	0.274	2013–1991
30	NVO	Novo Nordisk A/S	Pharmaceuticals	0.205	0.291	2013–1990
31	^FVX	5 Year Treasury yield	Risk-free rate	0.045	0.001	2013–1990
32	^IXIC	NASDAQ	Index	0.087	0.244	2013–1990
33	GSPC	S&P 500 index	Index	0.067	0.185	2013–1990

Table 5 Composition of unconstrained Maximum Sharpe and information ratio portfolios

Ticker	AAPL	KO	ACT	NVO	AMZN	GS	BLK	XOM	GE	IBM	Total:
Max *S*, ψ in %	19.13	0.78	37.99	42.1	0	0	0	0	0	0	100
Max *I*, ψ in %	18.51	0.7	17.81	20.04	4.49	2.52	8.38	13.67	1.72	12.17	100

Table 6 Composition of constrained Maximum Sharpe and information ratio portfolios

Ticker	AAPL	KO	ACT	NVO	AMZN	GS	BLK	BA	NOC	LMT	IBM	Total:
Max *S*, ψ in %	15.13	3.33	24.88	30	3.33	3.33	6.67	3.33	3.34	3.33	3.33	100
Max *I*, ψ in %	20.96	3.33	12.97	23.65	3.33	3.33	6.67	3.33	3.34	2.11	3.33	100 %
	MSFT	INTC	SAP	ORCL	LUV	GE	VZ	XOM				
	2.23	1.00	1.28	1.28	1.22	0.56	2.72	3.33				

References

Bacon, C. R. (2012). *Practical Risk-Adjusted Performance Measurement.* s.l.: Wiley.

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal, 10*, 28–43.

Dhrymes, P. J., Friend, I., & Gultekin, N. B. (1984). A critical reexamination of the empirical evidence on the arbitrage pricing theory. *The Journal of Finance, 39*(2), 323–346.

Fama, E. F., & French, K. R. (1996). Multifactor explanation of asset pricing anomalies. *The Journal of Finance, 51*(1), 55–84.

Flavin, T., & Wickens, M. R. (2002). *Macroeconomic Influences on Optimal Asset Allocation,* s.l.: s.n.

Goldfarb, A. D., & Idnani, A. (1983). A numerically stable dual method for solving strictly convex quadratic programs. *Mathematical Programming, 27*, 1–33.

Jensen, M. C. (1968). The performance of mutual funds in the period 1945–1964. *The Journal of Finance, 23*(2), 389–416.

Jorion, P. (1991). Bayesian and CAPM estimators of the means: Implications for portfolio selection. *Journal of Banking & Finance, 15*(3), 717–727.

Ledoit, O., & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance, 10*, 603–621.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics, 47*(1), 13–37.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*, 77–91.

Merton, R. C. (1980). On estimating the expected return on the market: an exploratory investigation. *Journal of Financial Economics, 8*, 1–39.

Michaud, R. O. (1989). The Markowitz optimization enigma: is 'Optimized' optimal? *Financial Analysts Journal, 45*(1), 31–42.

Puelz, A. V. (2001). Value-at-risk Based Portfolio Optimization. *Stochastic Optimization: Algorithms and Applications,* pp. 279–302

Sharpe, W. F. (1963). A simplified model for portfolio analysis. *Management Science, 9*(2), 277–293.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance, 19*(3), 425–442.

Xiao-Li, M., & Rubin, D. B. (1993). Maximum likelihood estimation via the ECM algorithm. *Biometrika, 80*(2), 267–278.

Yilmaz, T. (2010). *Improving Portfolio Optimization by DCC And DECO GARCH: Evidence from Istanbul Stock Exchange,* s.l.: s.n.