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COINTEGRATION FOR PERIODICALLY INTEGRATED PROCESSES

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Integration for seasonal time series can take the form of seasonal periodic or nonperiodic integration. When seasonal time series are periodically integrated, we show that any cointegration is either full periodic cointegration or full nonperiodic cointegration, with no possibility of cointegration applying for only some seasons. In contrast, seasonally integrated series can be seasonally, periodically or nonperiodically cointegrated, with the possibility of cointegration applying for a subset of seasons. Cointegration tests are analyzed for periodically integrated series. A residual-based test is examined, and its asymptotic distribution is derived under the null hypothesis of no cointegration. A Monte Carlo analysis shows good performance in terms of size and power. The role of deterministic terms in the cointegrating test regression is also investigated. Further, we show that the asymptotic distribution of the error-correction test for periodic cointegration derived by Boswijk and Franses (1995, *Review of Economics and Statistics* 77, 436–454) does not apply for periodically integrated processes.

1. INTRODUCTION

To date, cointegration analyses of long-run relationships in seasonal time series have been conducted primarily in terms of the separate (zero and seasonal frequency) unit roots implied by the seasonal differencing filter, which leads to the concept known as seasonal cointegration; see Hylleberg, Engle, Granger, and Yoo (1990), Engle, Granger, Hylleberg, and Lee (1993), Lee (1992), Johansen and Schaumburg (1999), and Cubadda (2001), among others. However, cointegration may also be considered season by season, and this route leads to so-called periodic cointegration, which is examined by Birchenhall,

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Bladen-Hovell, Chui, Osborn, and Smith (1989), Franses and Kloek (1995), Boswijk and Franses (1995), and others.

Seasonal cointegration can apply only for seasonally integrated (*SI*) processes, which are nonstationary processes that are made stationary by the application of annual differencing. In an analogous way, periodic cointegration can apply for periodically integrated (*PI*) processes, which are nonstationary but rendered stationary by application of a seasonally varying quasi-difference filter. In an *SI* process, nonstationary unit root behavior exists not only at the long-run (or zero) frequency but also at all the seasonal frequencies. Although this is not always discussed, the implication of these seasonal unit roots is that the seasons of the year are not cointegrated with each other and hence in principle "summer may become winter";[1] see, for example, Osborn (1991) and Ghysels and Osborn (2001). From an economic perspective, this implication may be unattractive. On the other hand, *PI* processes may be more plausible than *SI* ones, because they allow for nonstationarity in conjunction with cointegration applying between the separate seasons of the year (Osborn, 1991; Franses, 1996).

Although there has been little analysis of seasonal versus periodic cointegration, Franses (1993, 1995) shows that these imply different parameter restrictions on the cointegrating relationships when *SI* processes are considered.[2] In other words, periodic cointegration can apply between *SI*, in addition to between *PI*, processes. Boswijk and Franses (1995) propose a Wald test for periodic cointegration in *SI* processes and derive its asymptotic distribution, which they assert also applies when the individual series are *PI*. However, the present paper shows that this test has a different asymptotic distribution under the null hypothesis when applied to *PI*, rather than *SI*, processes. Indeed, because quarterly *PI* and *SI* processes differ in that the former implies one underlying unit root process across the four seasons whereas the latter implies four distinct unit root processes, we might anticipate that these cases will give rise to different asymptotic distributions.

Despite the availability of some theoretical results, the distribution of test statistics for periodic cointegration in *PI* processes is not fully resolved. A full dynamic system approach, in which equations are estimated jointly for observations relating to each season, can theoretically be applied (see, e.g., Ghysels and Osborn, 2001, pp. 171–176; Kleibergen and Franses, 1999). However, available sample sizes may make this feasible in practice only where data of a relatively high frequency are available, as in the application of Haldrup, Hylleberg, Pons, and Sansó (2007). Although a two-step approach of the Engle and Granger (1987) type has been used (e.g., Birchenhall et al., 1989; Franses and Kloek, 1995), the asymptotic distribution of the test statistic has not been derived for the case where all seasons are considered simultaneously. Franses (1996, p. 182) proposes testing for periodic cointegration through the application of the Boswijk and Franses (1996) *PI* test to the first-stage residuals and speculates as to its asymptotic distribution. The present paper contributes to this strand of literature by establishing that this test statistic follows the Phillips and

Ouliaris (1990) distribution, which enables asymptotically valid inference to be undertaken.

Prior to deriving the distributions of the test statistics for *PI* processes, Section 2 discusses the cointegration possibilities for these processes, which extends the discussion in Ghysels and Osborn (2001, pp. 168–171) and Osborn (2002). When the series are *PI*, we show that the only cointegration possibilities are periodic cointegration or nonperiodic cointegration, with cointegration for any one season implying cointegration for all seasons. The section also compares this to the wider set of possibilities for *SI* processes. Section 3 then derives the asymptotic distribution of the residual-based cointegration test for *PI* processes, which is followed (Section 4) by an analysis of the asymptotic distribution of the Boswijk and Franses (1995) cointegration test when applied to uncorrelated first-order *PI* processes. A Monte Carlo analysis in Section 5 examines the finite-sample distribution of the residual-based test, including an analysis of the role of deterministic terms in the regression, with a concluding section completing the paper. Proofs of our results appear in the Appendix.

2. PERIODIC INTEGRATION AND COINTEGRATION

After outlining the properties of *PI* processes, we discuss cointegration for *PI* and *SI* processes in Sections 2.2 and 2.3, respectively.

2.1. *PI* Processes

Consider a univariate time series $x_{s\tau}$, where the first subscript refers to the season (s) and the second subscript to the year (τ). For simplicity of exposition, we assume data are quarterly and that observations are available for precisely N years, so that the total sample size is $T = 4N$, with initial values $x_{10} = x_{20} = x_{30} = x_{40} = 0$. The annual difference operator is $\Delta_4 = 1 - L^4$, where L is the usual lag operator that works on the seasons ($L^k x_{s\tau} = x_{s-k,\tau}$). Note that, throughout the paper, it is understood that $x_{s-k,\tau} = x_{4-k+s,\tau-1}$ for $s - k \leq 0$.

Applications of periodic processes within economics have focused on autoregressive (AR) processes, with the pth-order zero-mean periodic AR, or PAR(p), process defined by[3]

$$x_{s\tau} = \phi_{1s} x_{s-1,\tau} + \phi_{2s} x_{s-2,\tau} + \cdots + \phi_{ps} x_{s-p,\tau} + e_{s\tau}, \qquad s = 1,2,3,4, \tag{1}$$

where $e_{s\tau}$ is white noise. Notice that all coefficients in (1) may vary with the season s. Although the conventional (nonperiodic) AR(p) process is a special case where $\phi_{is} = \phi_i$ (s = 1, 2, 3, 4) for all $i = 1, 2, \ldots, p$, our interest is periodic processes, that is, cases where at least some of these nonperiodic parameter restrictions do not hold.

Under the assumption that $x_{s\tau}$ is integrated of order one, and using a similar notation to that of Boswijk and Franses (1996), (1) can also be written as

$$(x_{s\tau} - \varphi_s x_{s-1,\tau}) = \psi_{1s}(x_{s-1,\tau} - \varphi_{s-1} x_{s-2,\tau}) + \cdots + \psi_{p-1,s}(x_{s-p+1,\tau} - \varphi_{s-p+1} x_{s-p,\tau}) + e_{s\tau} \tag{2}$$

with $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$. In the special case $\varphi_s = 1$ $(s = 1,2,3,4)$, (2) is a periodic $I(1)$ process, so that the first difference is a stationary PAR$(p-1)$ process. On the other hand, when $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$ but not all $\varphi_s = 1$ $(s = 1,2,3,4)$, (2) is a periodically integrated, or $PI(1)$, process with the quasi-difference $x_{s\tau} - \varphi_s x_{s-1,\tau}$ being stationary; see Ghysels and Osborn (2001, pp. 153–155) for further discussion of these cases. In the latter case $x_{s\tau} - \varphi_s x_{s-1,\tau}$ may have constant coefficients over seasons, although for convenience we refer to it as a stationary PAR process.

In both the $I(1)$ and $PI(1)$ cases, nonstationarity arises from a single common trend shared by the four quarterly observations of the time series; equivalently, there are three cointegration relationships between the quarters. It is convenient to explore this through the representation referred to as the vector of quarters (VQ) representation by Franses (1994), which is based on the vector $X_\tau = (x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$ and disturbance process $E_\tau = (e_{1\tau}, e_{2\tau}, e_{3\tau}, e_{4\tau})'$. Corresponding to (1), the VQ representation has the form

$$\Phi_0 X_\tau = \Phi_1 X_{\tau-1} + \Phi_2 X_{\tau-2} + \cdots + \Phi_P X_{\tau-P} + E_\tau,$$

where $P = [(p-1)/4] + 1$ and [.] indicates the integer part of the expression in brackets. Corresponding to the factorization in (2), the VQ representation can be written as

$$(\bar{\Phi}_0 - \bar{\Phi}_1 L^4) X_\tau = \Psi(L^4)^{-1} E_\tau = U_\tau, \tag{3}$$

where

$$\bar{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\varphi_2 & 1 & 0 & 0 \\ 0 & -\varphi_3 & 1 & 0 \\ 0 & 0 & -\varphi_4 & 1 \end{bmatrix}, \qquad \bar{\Phi}_1 = \begin{bmatrix} 0 & 0 & 0 & \varphi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the determinant $|\bar{\Phi}_0 - \bar{\Phi}_1 z| = 0$ has a single unit root, while the stationary vector process $\Psi(L^4) U_\tau = E_\tau$ captures the stationary PAR$(p-1)$ process in $u_{s\tau} = x_{s\tau} - \varphi_s x_{s-1,\tau}$ embodied in (2).

From (3), the annual difference series $\Delta_4 X_\tau = X_\tau - X_{\tau-1}$ can be written as

$$\Delta_4 X_\tau = (\Theta_0 - \Theta_1 L^4) U_\tau = C(L^4)\Psi(L^4)^{-1} E_\tau, \tag{4}$$

where the vector process $C(L^4)U_\tau = (\Theta_0 - \Theta_1 L^4)U_\tau$ has three noninvertible unit roots, which reflect the overdifferencing inherent in annual differencing for a process with a single AR unit root. Therefore, $C(1)$ is of rank one, and it is possible to write

$$C(1) = \Theta_0 - \Theta_1 = ab', \tag{5}$$

where $a = (1, \varphi_2, \varphi_2\varphi_3, \varphi_2\varphi_3\varphi_4)'$, $b = (1, \varphi_1\varphi_3\varphi_4, \varphi_1\varphi_4, \varphi_1)'$.

An implication of (4) and (5) is that the four elements of X_τ share a single common stochastic trend, namely, $b'\Psi(1)^{-1}\sum_{t=1}^{\tau} E_t = b'\sum_{t=1}^{\tau} U_t$, to which they adjust with periodic adjustment coefficients given by the elements of a. Further details can be found in Boswijk and Franses (1996).

2.2. Cointegration for *PI* Processes

Now consider the $m \times 1$ vector process $x_{s\tau} = [x_{s\tau}^{(1)}, \ldots, x_{s\tau}^{(m)}]'$ in which each element is either a $PI(1)$ or an $I(1)$ process. That is,

$$x_{s\tau}^{(j)} = \varphi_s^{(j)} x_{s-1,\tau}^{(j)} + u_{s\tau}^{(j)} \quad \text{with} \prod_{s=1}^{4} \varphi_s^{(j)} = 1, \qquad s = 1,2,3,4, \qquad j = 1,\ldots,m, \tag{6}$$

in which each $u_{s\tau}^{(j)}$ is a stationary process (which is, in general, periodic)

$$(1 - \psi_{1s}^{(j)} L - \cdots - \psi_{p-1,s}^{(j)} L^{p-1}) u_{s\tau}^{(j)} = e_{s\tau}^{(j)}$$

with $E_{s\tau} = (e_{s\tau}^{(1)}, \ldots, e_{s\tau}^{(m)})'$ vector white noise and $E[E_{s\tau}E_{s\tau}'] = \Sigma$ positive definite.[4] Therefore, we can define the VQ representation as in (3) for each $X_\tau^{(j)} = (x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)})'$, $j = 1, \ldots, m$.

Cointegration can then be defined as follows.

DEFINITION 1. *The $m \times 1$ vector $x_{s\tau} = [x_{s\tau}^{(1)}, \ldots, x_{s\tau}^{(m)}]'$ of periodic processes satisfying (6) is periodically cointegrated if there exist $m \times r$ matrices α_s of rank r such that the linear combinations $\alpha_s' x_{s\tau}$ are (periodically) stationary for each $s = 1, \ldots, 4$.*

Although it was not formally defined in this way, the idea of periodic cointegration appears to have been applied first by Birchenhall et al. (1989). Notice that nonperiodic cointegration, with $\alpha_s = \alpha$ for $s = 1, 2, 3, 4$, is permitted here as a special case of periodic cointegration.

In their discussion of periodic cointegration, Boswijk and Franses (1995) distinguish full and partial periodic cointegration, where the former corresponds to Definition 1 and the latter to the situation where stationary linear combinations $\alpha_s' x_{s\tau}$ exist for only some $s = 1, \ldots, 4$. However, Ghysels and Osborn

(2001) show that partial periodic cointegration cannot apply between two $PI(1)$ processes; such processes are either (fully) periodically cointegrated or no cointegrating relationship exists for any $s = 1, \ldots, 4$. This result is generalized in Lemma 1 for the case of m $PI(1)$ processes.

LEMMA 1. *Consider the* $m \times 1$ *vector* $x_{s\tau} = [x_{s\tau}^{(1)}, \ldots, x_{s\tau}^{(m)}]'$ *of periodic processes in (6), such that the* $m \times r$ *matrix* α_s *of rank r defines all linearly independent stationary linear combinations* $\alpha_s' x_{s\tau}$ *for some* $s = 1, \ldots, 4$. *Then*

- *(i)* α_s, *together with the coefficients* $\varphi_s^{(j)} (s = 1,2,3,4; j = 1, \ldots, m)$ *of (6), determine the* $m \times r$ *matrix* α_q *of rank r, which must exist for each* $q = 1,2,3,4$, $q \neq s$ *such that* $\alpha_q' x_{q\tau}$ *is stationary;*
- *(ii) nonperiodic cointegration with* $\alpha_s = \alpha$ $(s = 1,2,3,4)$ *applies if and only if* $\varphi_s^{(j)} = \varphi_s$, $j = 1, \ldots, m$ *in (6).*

The first part of Lemma 1 implies that there must be the same number of cointegrating relationships between $PI(1)$ processes for all seasons s = 1, 2, 3, 4. Thus, as in the bivariate case considered by Ghysels and Osborn (2001) and Osborn (2002), partial periodic cointegration cannot apply between $PI(1)$ processes. Further, given the cointegrating vectors that apply for one season and the univariate *PI* coefficients of (1), then all four sets of cointegrating relations can be determined. It is worth noting that part (i) of Lemma 1 is implicit in the results of Kleibergen and Franses (1999), but they do not draw it out from their analysis.

Part (ii) of the lemma further establishes that the same (nonperiodic) cointegrating relations can apply over seasons if and only if all processes have identical univariate *PI* coefficients. The proof of this lemma rests on the fact that the VQ process corresponding to a $PI(1)$ variable is driven by a single unit root process. The stationary relationships between observations for the seasons that exist for each $X_\tau^{(j)} = [x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)}]'$ then imply that cointegrating relations for the vector $x_{s\tau}$ can be mapped from season to season.

Conventional cointegration between $I(1)$ processes (which may be periodic) provides a special case of Lemma 1, where the same cointegrating relations apply for all seasons (quarters) of the year and all $\varphi_s^{(j)}$ of (6) are unity.

2.3. Cointegration for *SI* Processes

Unlike the quasi-differencing $x_{s\tau} - \varphi_s x_{s-1,\tau}$ of (2) with $\varphi_1 \varphi_2 \varphi_3 \varphi_4 = 1$ that renders the $PI(1)$ process stationary, $SI(1)$ processes (that contain a unit root at the zero and each seasonal frequency) are made stationary and invertible by annual differencing. Such processes contain four unit roots, implying that the quarters of the year are not cointegrated with each other; Osborn (1991) and Franses (1994) provide discussions of some of the implications. In the context of cointegration between elements of an $m \times 1$ vector $x_{s\tau}$, if each element is

$SI(1)$ then the lack of cointegration across the seasons implies that distinct cointegrating relations can apply for each of the vectors $x_{s\tau}$ for $s = 1,2,3,4$. This is the essence of Lemma 2.

LEMMA 2. *Consider the* $m \times 1$ *vector* $x_{s\tau} = [x_{s\tau}^{(1)}, \ldots, x_{s\tau}^{(m)}]'$ *of* $SI(1)$ *processes. Then the existence of an* $m \times r$ *matrix* α_s *of rank r such that* $\alpha_s' x_{s\tau}$ *is stationary for some* $s = 1, \ldots, 4$ *has no implications for the existence or nature of cointegration across the elements of* $x_{q\tau}$ *for* $q \neq s$.

An immediate consequence of Lemma 2 is that full and partial periodic cointegration are possibilities for $SI(1)$ processes.

So-called seasonal cointegration, which corresponds to cointegration at the distinct seasonal spectral frequencies, is another possibility for *SI* processes and is analyzed by Engle et al. (1993), Cubadda (2001), Johansen and Schaumburg (1999), and Lee (1992). However, our analysis focuses on testing for periodic cointegration. More specifically, we are particularly interested in testing for cointegration for *PI* processes. However, the case of *SI* processes is relevant, because Boswijk and Franses (1995) claim that the same asymptotic distribution results when their test is applied to both *SI* and *PI* processes.

3. RESIDUAL-BASED TEST FOR PERIODIC COINTEGRATION

This section analyzes the periodic analogue of the Engle and Granger (1987) test, which applies a test for periodic integration to the residuals from a first-stage regression involving nonstationary $PI(1)$ variables. We first set out the test regression, and then, before obtaining the distribution of the test statistic in Section 3.3, Section 3.2 examines the properties of the vector $x_{s\tau} = [x_{s\tau}^{(1)}, \ldots, x_{s\tau}^{(m)}]'$ in the absence of cointegration.

3.1. The Test Regression

As usual, a residual-based test requires that the potential cointegrating relationship being examined is unique. That is, either there exists at most one cointegrating vector or, if there potentially exist $1 < r < m$ cointegrating vectors between the separate series, then (exclusion) restrictions are imposed to ensure uniqueness. From the analysis of the previous section, we know that cointegration applying for one season between *PI* processes implies cointegration for all seasons. Therefore, it is anticipated that efficiency gains will result by considering all seasons jointly.

To keep notation simple, we assume that only one cointegrating relationship may exist across the m variables. Arbitrarily normalizing on the first element of x, we propose fitting the periodic regression

$$x_{s\tau}^{(1)} = \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + v_{s\tau}, \qquad s = 1,2,3,4 \tag{7}$$

and then applying the periodic integration test of Boswijk and Franses (1996) to the residuals $\hat{v}_{s\tau} = x_{s\tau}^{(1)} - \sum_{i=2}^{m} \hat{\beta}_{is} x_{s\tau}^{(i)}$. The intuition is that, in the absence of cointegration, the residuals $\hat{v}_{s\tau}$ follow a nonstationary *PI* process (see Franses, 1996, pp. 181–182).

Now, partition $x_{s\tau}$ as

$$x_{s\tau} = (x_{s\tau}^{(1)}, z'_{s\tau})', \qquad z'_{s\tau} = (x_{s\tau}^{(2)}, \ldots, x_{s\tau}^{(m)})' \tag{8}$$

so that $z_{s\tau}$ comprises the vector of right-hand-side variables in (7). We assume that all $j = 1, \ldots, m$ variables are $PI(1)$ processes, as in (6), with the variance-covariance disturbance matrix corresponding to the system being

$$E[E_{s\tau} E'_{s\tau}] = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma'_{1z} \\ \sigma_{1z} & \Sigma_{zz} \end{bmatrix}, \tag{9}$$

where Σ in (9) is partitioned conformably with $x_{s\tau}$ in (8).

3.2. Properties of the *PI* System

As discussed in Section 2.1, the matrix $C^{(j)}(L^4)$ in the representation of (4) for an annually differenced *PI* process has three unit roots, and hence

$$C^{(j)}(1) = (\Theta_0^{(j)} - \Theta_1^{(j)}) = a^{(j)} b^{(j)\prime}, \qquad j = 1, \ldots, m, \tag{10}$$

where $a^{(j)} = [1,\ \varphi_2^{(j)},\ \varphi_2^{(j)}\varphi_3^{(j)},\ \varphi_2^{(j)}\varphi_3^{(j)}\varphi_4^{(j)}]'$, $b^{(j)} = [1,\ \varphi_1^{(j)}\varphi_3^{(j)}\varphi_4^{(j)},\ \varphi_1^{(j)}\varphi_4^{(j)},\ \varphi_1^{(j)}]'$.

Stacking the processes, and using the annual difference representation of (4), yields

$$\Delta_4 X_\tau = \Theta_0^x U_\tau^x - \Theta_1^x U_{\tau-1}^x = (\Theta_0^x - \Theta_1^x L^4)\Psi^x(L^4)^{-1} E_\tau^x, \tag{11}$$

where $\Delta_4 X_\tau = (\Delta_4 X_\tau^{(1)\prime}, \Delta_4 X_\tau^{(2)\prime}, \ldots, \Delta_4 X_\tau^{(m)\prime})'$, $\Delta_4 X_\tau^{(j)} = (\Delta_4 x_{1\tau}^{(j)}, \Delta_4 x_{2\tau}^{(j)}, \Delta_4 x_{3\tau}^{(j)}, \Delta_4 x_{4\tau}^{(j)})'$, with corresponding definitions for E_τ^x and U_τ^x, while $\Psi^x(L^4)$ is a block diagonal matrix with jth block equal to $\Psi^{(j)}(L^4)$ containing the stationary (periodic) AR coefficients for series j. Similarly, the moving average (MA) coefficient matrices in (11) are block diagonal, of the form

$$\Theta_i^x = \begin{bmatrix} \Theta_i^{(1)} & 0 & \ldots & 0 \\ 0 & \Theta_i^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Theta_i^{(m)} \end{bmatrix}, \qquad i = 0, 1,$$

and

$$E[E_\tau^x E_\tau^{x\prime}] = \Sigma \otimes I_4.$$

The long-run covariance matrix between the $m \times 4$ processes in X_τ is then given by (see also Boswijk and Franses, 1995, p. 440)

$$\begin{aligned}\Omega &= \sum_{i=-\infty}^{\infty} E[\Delta_4 X_\tau \Delta_4 X_{\tau-i}'] \\ &= \sum_{i=-\infty}^{\infty} E[\Theta_0^x U_\tau^x - \Theta_1^x U_{\tau-1}^x][\Theta_0^x U_{\tau-i}^x - \Theta_1^x U_{\tau-i-1}^x]' \\ &= [\Theta_0^x - \Theta_1^x]\Psi^x(1)^{-1}(\Sigma \otimes I_4)\Psi^x(1)'^{-1}[\Theta_0^x - \Theta_1^x]'. \end{aligned} \quad \textbf{(12)}$$

In the absence of cointegration between the $x_{s\tau}^{(j)}$ $(j = 1, 2, \ldots, m)$, cointegration applies only across the seasons within each $x_{s\tau}^{(j)}$, and we have

$$C^x(1) = \Theta_0^x - \Theta_1^x = a^x b^{x\prime}, \quad \textbf{(13)}$$

where the $4m \times m$ matrices a^x, b^x are defined by (in an obvious notation)

$$a^x = \begin{bmatrix} a^{(1)} & 0 & \ldots & 0 \\ 0 & a^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^{(m)} \end{bmatrix}, \qquad b^x = \begin{bmatrix} b^{(1)} & 0 & \ldots & 0 \\ 0 & b^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b^{(m)} \end{bmatrix} \quad \textbf{(14)}$$

and all submatrices in (14) are 4×1. Thus, $C^x(1)$ is of rank m.

However, if cointegration exists across processes, then $C^x(1)$ is of rank $k < m$, and hence a^x and b^x do not have the block diagonal form of (14). Specifically, a^x and b^x are then matrices of rank k, with dimension $4m \times k$.

Returning to the case of no cointegration, Lemma 3, which follows, establishes the asymptotic distribution of the scaled vector $X_\tau = [X_\tau^{(1)\prime}, Z_\tau']'$ relevant for the regression (7). The result is obtained by accounting for the contemporaneous correlation between the disturbances through the decomposition $\Sigma = PP'$ where P is upper triangular.

LEMMA 3. *Consider the vector of m* $PI(1)$ *processes defined in (6), (8), (9), and (11), with no cointegration applying across the m processes. Also define the* $4m \times 1$ *vector Brownian motion* $W^x(r)$ *with covariance matrix* I_{4m}, *where* $W^x(r) = [W^{(1)}(r)', W^z(r)']'$ *in which* $W^{(1)}(r)$ *is* 4×1, $W^z(r)$ *is* $4n \times 1$, *and* $n = m - 1$. *Then, as* $N = T/4 \to \infty$,

$$\frac{1}{\sqrt{N}} X_{[rN]} = \frac{1}{\sqrt{N}} \begin{bmatrix} X^{(1)}_{[rN]} \\ Z_{[rN]} \end{bmatrix} \Rightarrow B(r) = \begin{bmatrix} B^{(1)}(r) \\ B^{z}(r) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{11}^{1/2} a^{(1)} b^{(1)\prime} \Psi^{(1)}(1)^{-1} \left(\sqrt{1-\rho_{1z}'\rho_{1z}} W^{(1)}(r) + (\rho_{1z}' \otimes I_4) W^{z}(r)\right) \\ a^{z} b^{z\prime} \Psi^{z}(1)^{-1} (P_{zz} \otimes I_4) W^{z}(r) \end{bmatrix}, \tag{15}$$

where $[rN]$ *is the integer part of* rN, $\rho_{1z} = \sigma_{11}^{-1/2} P_{zz}^{-1} \sigma_{1z}$, *the upper triangular matrix* P_{zz} *satisfies* $P_{zz}P_{zz}' = \Sigma_{zz}$, *and* a^{z}, b^{z} *are the lower right-hand* $4n \times n$ *blocks of* a^{x}, b^{x}, *respectively, in (14).*

Here and throughout the paper $\Rightarrow$ indicates convergence in distribution. Note that we can define standard Brownian motions underlying (15) as

$$\begin{aligned} \widetilde{w}^{(1)}(z) &= (b^{(1)\prime}\Psi^{(1)}(1)^{-1}\Psi^{(1)}(1)^{\prime -1} b^{(1)})^{-1/2} b^{(1)\prime}\Psi^{(1)}(1)^{-1} \\ &\quad \times \left(\sqrt{1-\rho_{1z}'\rho_{1z}} W^{(1)}(r) + (\rho_{1z}' \otimes I_4) W^{z}(r)\right) \\ \widetilde{w}^{(j)}(r) &= (b^{(j)\prime}\Psi^{(j)}(1)^{-1}\Psi^{(j)}(1)^{\prime -1} b^{(j)} p^{(j)} p^{(j)\prime})^{-1/2} b^{(j)\prime}\Psi^{(j)}(1)^{-1} \\ &\quad \times (p^{(j)} \otimes I_4) W^{z}(r), \qquad j = 2,3,\ldots,m, \end{aligned} \tag{16}$$

where $p^{(j)}$ is the $(j-1)$th row of P_{zz}. Then, from (16), we can write

$$\begin{aligned} B_s^{(1)}(r) &= \omega_1 a_s^{(1)} \widetilde{w}^{(1)}(r), \\ B_s^{(j)}(r) &= w_j a_s^{(j)} \widetilde{w}^{(j)}(r), \qquad j = 2,3,\ldots,m, \end{aligned} \tag{17}$$

in which

$$\begin{aligned} \omega_1 &= \sigma_{11}^{0.5} (b^{(1)\prime}\Psi^{(1)}(1)^{-1}\Psi^{(1)}(1)^{\prime -1} b^{(1)})^{0.5}, \\ \omega_j &= (p^{(j)} p^{(j)\prime} b^{(j)\prime}\Psi^{(j)}(1)^{-1}\Psi^{(j)}(1)^{\prime -1} b^{(j)})^{0.5}, \qquad j = 2,3,\ldots,m. \end{aligned}$$

Therefore, the scalar standard Brownian motions $\widetilde{w}^{(j)}(r)$ in (16) can be thought of as the stochastic trends underlying the m individual *PI* processes, which in turn derive from the elements of the standard vector Brownian motion $W^{x}(r)$.

It is clear from (15) or (16) that, in general, Brownian motions processes relating to $x^{(1)}$ and z are correlated. That is, when the contemporaneous covariance σ_{1z} in (9) is nonzero, $W^{z}(r)$ influences $\widetilde{w}^{(1)}(r)$. This effect disappears in the special case of $\sigma_{1z} = 0$, because $\rho_{1z} = 0$ when $x^{(1)}$ is uncorrelated with $x^{(2)}, \ldots, x^{(m)}$.

3.3. Asymptotic Distribution of the Test Statistic

We now turn to the properties of the residuals resulting from ordinary least squares (OLS) estimation of (7), which are summarized in Lemma 4.

LEMMA 4. *Consider the vector of* $PI(1)$ *processes defined in (6), (8), (9), and (11), with no cointegration applying across the m processes. The* 4×1 *vector* $\hat{V}_\tau = [\hat{\nu}_{1\tau}, \hat{\nu}_{2\tau}, \hat{\nu}_{3\tau}, \hat{\nu}_{4\tau}]'$ *of residuals from (7) for year* τ *then satisfies, as* $N = T/4 \to \infty$,

$$\frac{1}{\sqrt{N}} \hat{V}_{[rN]} \Rightarrow l_{11} \sigma_{11}^{1/2} \omega a^{(1)} \overline{w_m}(r), \tag{18}$$

where $[rN]$ *is the integer part of rN,* l_{11} *is a scalar,* $\omega = (b^{(1)\prime} \Psi^{(1)}(1)^{-1} \Psi^{(1)}(1)^{\prime -1} b^{(1)})^{0.5}$, $a^{(1)} = [1, \varphi_2^{(1)}, \varphi_2^{(1)} \varphi_3^{(1)}, \varphi_2^{(1)} \varphi_3^{(1)} \varphi_4^{(1)}]'$, $b^{(1)} = [1, \varphi_1^{(1)} \varphi_3^{(1)} \varphi_4^{(1)}, \varphi_1^{(1)} \varphi_4^{(1)}, \varphi_1^{(1)}]'$, *and the univariate Brownian motion* $\overline{w_m}(r)$ *is defined by*

$$\overline{w_m}(r) = \overline{w^{(1)}}(r) - \int \overline{w^{(1)}}(r) \overline{W^z}(r)' \, dr \left[\int \overline{W^z}(r) \overline{W^z}(r)' \, dr \right]^{-1} \overline{W^z}(r),$$

$$= k' \overline{W}^x(r), \tag{19}$$

in which $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ *is* $m \times 1$ *standard Brownian motion with covariance matrix* I_m *and* $k' = [1, -\int \overline{w^{(1)}}(r) \overline{W^z}(r)' \, dr (\int \overline{W^z}(r) \overline{W^z}(r)' \, dr)^{-1}]$.

Lemma 4 implies that the residuals from (7) asymptotically retain the same nonstationary periodic coefficients as the univariate process for $x_{s\tau}^{(1)}$ in (6). This is easily seen by comparing (18) with the first equation of (17).

Building on the implication of Lemma 4 that the residuals of (7) retain the *PI* properties of $x_{s\tau}^{(1)}$ in (6), the strategy of testing for periodic integration in the residuals of (7) is clear. More specifically, following Franses (1996, pp. 181–182), we propose testing the periodic integration null hypothesis $\varphi_1 \varphi_2 \varphi_3 \varphi_4 = 1$ against the alternative $\varphi_1 \varphi_2 \varphi_3 \varphi_4 < 1$.[5] The unrestricted model is the PAR(p) regression

$$\hat{\nu}_{s\tau} = \sum_{j=1}^{p} \phi_{js} \hat{\nu}_{s-j,\tau} + \varepsilon_{s\tau}, \qquad s = 1,2,3,4. \tag{20}$$

Under the alternative hypothesis, the residuals of (20) are stationary, implying that either periodic cointegration or nonperiodic cointegration exists between the processes for all seasons s = 1, 2, 3, 4. Under the null hypothesis the residuals $\hat{\nu}_{s\tau} \sim PI(1)$, so that there is no cointegration between the *PI* processes. In this case, based on (6) and following Boswijk and Franses (1996), (20) can be reparameterized as

$$\hat{\upsilon}_{s\tau} = \varphi_s \hat{\upsilon}_{s-1,\tau} + \sum_{j=1}^{p-1} \psi_{js} (\hat{\upsilon}_{s-j,\tau} - \varphi_{s-j} \hat{\upsilon}_{s-j-1,\tau}) + \varepsilon_{s\tau}, \qquad s = 1,2,3,4 \tag{21}$$

with the restriction $\varphi_1 \varphi_2 \varphi_3 \varphi_4 = 1$ imposed, but ψ_{js} unrestricted, with estimation achieved using nonlinear least squares.

We employ the test of periodic integration proposed by Boswijk and Franses (1996), which uses the likelihood ratio (LR) statistic

$$LR = N \ln(\tilde{\sigma}_0^2 / \tilde{\sigma}^2), \tag{22}$$

where $\tilde{\sigma}^2$ is the unrestricted maximum likelihood estimator of $\mathrm{Var}(\varepsilon_{s\tau})$ in (20) and $\tilde{\sigma}_0^2$ is the corresponding estimator from (21) when the restriction $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$ is imposed.

Theorem 1 establishes the asymptotic distribution of this periodic cointegration test statistic.

THEOREM 1. *Under the null hypothesis of no cointegration between the PI processes of (6), (8), (9), and (11), the LR test statistic of (22) applied to the residuals from (7) using the* PAR(p) *regression (20)–(21) has asymptotic distribution*

$$LR \Rightarrow (k'k)^{-1} \left\{ \int \overline{w_m}(r)^2\, dr \right\}^{-1} \left\{ \int \overline{w_m}(r) d\overline{w_m}(r) \right\}^2, \tag{23}$$

where $\overline{w_m}(r)$ *is defined in (19) and k in Lemma 4.*

The distribution of the test statistic in (23) is the square of the Dickey–Fuller test for cointegration using the residuals of a (nonperiodic) regression, as derived by Phillips and Ouliaris (1990). It is clear from (19) that this asymptotic distribution depends on the number of regressors in (7), namely, $n = m - 1$. Consequently, the distribution of the LR test statistic in (23) also depends on m. However, the distribution is invariant to the values of the *PI* coefficients for the processes in (6) and the disturbance covariances of (9).

Also in common with Phillips and Ouliaris (1990), the presence of stationary autocorrelation plays no role in the asymptotic distribution. However, in this periodic case, the augmentation in the test regression of (20)–(21) is periodic in form, because the possibility of stationary PAR dynamics is permitted under the null hypothesis.

4. THE BOSWIJK AND FRANSES TEST

Boswijk and Franses (1995) propose Wald tests for periodic cointegration relating to a specific season s and over all seasons with an error-correction mechanism (ECM) framework. This test is built on the cointegration test of Boswijk (1994), which was developed in a nonperiodic context. To avoid issues Boswijk and Franses encounter concerned with the possible dependence of some asymptotic distributions on correlation between the disturbances of the processes considered, we confine our attention to the "spurious regression" case where the variables are mutually uncorrelated. Also for simplicity, we continue to assume that all variables have zero means, with no deterministic terms included in the estimated ECM model, and here consider only PAR(1) processes. Although this is a special case, it is sufficient to establish that, contrary to the statement of Boswijk and Franses (1995), the asymptotic distribution of their test applied to *PI* processes differs from the result they obtain for *SI* processes.

Using the notation of the previous section, and arbitrarily assuming that the first variable of $x_{s\tau}$ is the dependent variable, the periodic ECM model is

$$\Delta_4 x_{s\tau}^{(1)} = \gamma_s(x_{s,\tau-1}^{(1)} - \kappa_s' z_{s,\tau-1}) + u_{s\tau}, \qquad s = 1,2,3,4,$$

where κ_s is an $n \times 1$ vector. This can also be written as

$$\Delta_4 x_{s\tau}^{(1)} = \delta_{0s} x_{s,\tau-1}^{(1)} + \delta_{1s}' z_{s,\tau-1} + u_{s\tau}, \qquad s = 1,2,3,4, \tag{24}$$

where $\delta_{0s} = \gamma_s$ and $\delta_{1s} = -\gamma_s \kappa_s$. With the assumption of uncorrelated PAR(1) processes, $u_{s\tau}^{(j)} = e_{s\tau}^{(j)}$ in (6), and $\sigma_{1z} = 0$ with Σ_{zz} being a diagonal matrix in (9). In comparison with Boswijk and Franses (1995), no conditioning is required in (24), because of our simplifying assumption that the variables are uncorrelated PAR(1) processes.

Using a similar notation to Boswijk and Franses (1995), the Wald statistic to test the null of no cointegration in season s, or equivalently to test $\delta_{0s} = 0$, $\delta_{1s} = 0$ in (24), is

$$Wald_s = \hat{\delta}_s'(\hat{\mathrm{Var}}[\hat{\delta}_s])^{-1}\hat{\delta}_s, \tag{25}$$

where $\hat{\delta}_s = (\hat{\delta}_{0s}, \hat{\delta}_{1s}')'$ is the OLS estimator of the relevant coefficients and $\hat{\mathrm{Var}}[\hat{\delta}_s]$ is the corresponding estimated OLS covariance matrix. When all seasons are considered, the joint cointegration test statistic for the null hypothesis $\delta = 0$, where $\delta = [\delta_1', \delta_2', \delta_3', \delta_4']'$, is given by (in an obvious notation)

$$Wald = \hat{\delta}'(\hat{\mathrm{Var}}[\hat{\delta}])^{-1}\hat{\delta} = \sum_{s=1}^{4} \hat{\delta}_s'(\hat{\mathrm{Var}}[\hat{\delta}_s])^{-1}\hat{\delta}_s. \tag{26}$$

Note that the Wald statistic in (26) is the sum of the individual $Wald_s$ of (25) because of the block orthogonality of the regressors when (24) is estimated using seasonal dummy variables.

As noted by Ghysels and Osborn (2001, pp. 176–179), the null distribution obtained by Boswijk and Franses (1995) assumes that $x_{s\tau}$ is a vector of *SI* processes. More specifically, Assumption 1 of Boswijk and Franses (1995, p. 440) does not require the long-run variance-covariance matrix Ω of the vector Brownian motion process corresponding to $(x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$ to be positive definite, which allows the possibility of one or more components being *PI* processes. However, the proof of their Theorem 2 assumes that $\tilde{C}_s'\Omega\tilde{C}_s$ is strictly positive.[6] Consequently, the asymptotic distributions derived by Boswijk and Franses require Ω to have full rank, ruling out the possibility that any element of $x_{s\tau}$ is periodically integrated.

Under the null hypothesis of no periodic cointegration, and assuming *SI* processes, Boswijk and Franses (1995) establish that the distribution of the $Wald_s$ statistic used to test for cointegration relating to an individual season s is identical to that obtained by Boswijk (1994) for the nonperiodic case. Theorem 2, which follows, shows that this result does not carry over to the case of $PI(1)$ processes. Indeed, for such processes, the theorem shows that the distribution of Boswijk (1994) emerges in relation to the test statistic for full periodic cointegration.

THEOREM 2. *Assuming that the PI processes of (6), (8), (9), and (11) are uncorrelated* PAR(1) *processes, the asymptotic distributions of the Wald test statistics proposed by Boswijk and Franses under the null hypothesis of no cointegration are given by*

(i) for the $Wald_s$ *test of* $\delta_{0s} = \delta_{1s} = 0$ *for an individual s*

$$Wald_s \Rightarrow 4\,\frac{(a_s^{(1)})^2}{a^{(1)\prime}a^{(1)}}\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'\,dr\right)^{-1} \times\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right); \quad \textbf{(27)}$$

(ii) for the joint Wald test of $\delta_{0s} = \delta_{1s} = 0,\ s = 1,2,3,4,$

$$Wald \Rightarrow 4\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'\,dr\right)^{-1} \times\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right), \quad \textbf{(28)}$$

where $\overline{W^x}(r)' = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ *is m-vector standard Brownian motion and* $a^{(1)} = [1, \varphi_2^{(1)}, \varphi_2^{(1)}\varphi_3^{(1)}, \varphi_2^{(1)}\varphi_3^{(1)}\varphi_4^{(1)}]'$*, which has sth element* $a_s^{(1)}$.

There are two important differences between the distributional results given in (27) and (28) and those of Boswijk and Franses (1995) for *SI* processes. First, the statistic in (27) does not follow the distribution of Boswijk (1994), because of the multiplicative factor $\lambda_s = 4(a_s^{(1)})^2/a^{(1)\prime}a^{(1)}$. Because these λ_s average unity over s = 1, 2, 3, 4, the scaling will inflate or deflate values relative to the Boswijk (1994) distribution, depending on the specific *PI* coefficients and the season *s*.

Second, the distribution defined by (28) is four times the distribution obtained by Boswijk (1994). Intuitively, this arises because there is only one underlying stochastic trend for each vector process $X_\tau^{(j)}$ and hence, as discussed in Section 2, there can be only one linearly independent cointegrating relationship over the four quarters of the year. Consequently, when the Wald test is applied to the $PI(1)$ variables, effectively a single cointegration relationship is being tested four times (once for each quarter).

The asymptotic distribution of (28) is not that derived by Boswijk and Franses (1995) for *SI* processes. To be specific, because an *SI* process for a quarterly series involves four distinct unit root processes, these are reflected in the asymptotic distribution. For uncorrelated *SI* processes, the asymptotic Boswijk–Franses distribution is (Ghysels and Osborn, 2001, p. 178)

$$Wald \Rightarrow \sum_{s=1}^{4} \left(\int W_s^x(r)\, dw_s^{(1)}(r) \right)' \left(\int W_s^x(r) W_s^x(r)' \right)^{-1} \left(\int W_s^x(r)\, dw_s^{(1)}(r) \right), \tag{29}$$

where $W_s^x(r) = [w_s^{(1)}(r), W_s^z(r)']'$ is formed by selecting elements of the $4m \times 1$ vector standard Brownian motion $W^x(r)$ corresponding to season s. It is obvious that (28) and (29) differ, with the former being four times the Boswijk (1994) distribution whereas the latter is the sum of four independent distributions of this type. Indeed, this comparison also clarifies the role played by the four distinct unit roots underlying an *SI* process and that therefore appear in (29) as against the single unit root underlying a *PI* process.

5. MONTE CARLO ANALYSIS

In this section we present a selection of Monte Carlo results relating to the empirical size and power of the residual-based test for periodic cointegration analyzed in Section 3. Section 5.1 considers zero-mean processes, with the analysis of Section 5.2 allowing the possibility of nonzero trends.

5.1. Zero-Mean Processes

We investigate empirical size[7] for zero-mean processes generated through the bivariate model, where $x_{s\tau} = (y_{s\tau}, z_{s\tau})'$, such that

$$\begin{aligned} y_{s\tau} &= \varphi_s^y y_{s-1,\tau} + u_{s\tau}^y, \qquad \prod_{s=1}^{4} \varphi_s^y = 1, \\ z_{s\tau} &= \varphi_s^z z_{s-1,\tau} + u_{s\tau}^z, \qquad \prod_{s=1}^{4} \varphi_s^z = 1, \end{aligned} \tag{30}$$

where the periodic components satisfy the univariate AR processes

$$u_{s\tau}^{(j)} = \psi_1 u_{s-1,\tau}^{(j)} + \psi_2 u_{s-2,\tau}^{(j)} + \psi_3 u_{s-3,\tau}^{(j)} + e_{s\tau}^{(j)}, \qquad j = y, z \tag{31}$$

and

$$E\left[\begin{bmatrix} e^y_{s\tau} \\ e^z_{s\tau} \end{bmatrix} \begin{bmatrix} e^y_{s\tau} & e^z_{s\tau} \end{bmatrix}\right] = \begin{bmatrix} 1 & \sigma_{yz} \\ \sigma_{yz} & 1 \end{bmatrix}, \qquad \sigma_{yz} = \{0.0, 0.4, 0.8\}. \tag{32}$$

The case of nonperiodic stationary components (31) is sufficient to illustrate the size and power properties of the test, with the processes being periodic through the *PI* coefficients in (30). Although results are presented only for stationary AR coefficients in (31) being common across the two processes, similar results have been obtained from simulations where these processes are distinct. Finally, it can be seen that (32) permits three levels of contemporaneous correlation between the innovations $e^y_{s\tau}$ and $e^z_{s\tau}$.

As shown in Lemma 4, the data generating process (DGP) of (30) implies that the residuals follow a *PI* process when the regression of (7) is estimated.

Empirical power is obtained from the DGP

$$\begin{aligned} y_{s\tau} &= \beta_s z_{s,\tau} + u_{s\tau}, \qquad (1-\psi L)(1-\psi_1^* L - \psi_2^* L^2 - \psi_3^* L^3) u_{s\tau} = e^y_{s\tau}, \\ z_{s\tau} &= \varphi^z_s z_{s-1,\tau} + e^z_{s\tau}, \qquad \prod_{s=1}^{4} \varphi^z_s = 1, \end{aligned} \tag{33}$$

focusing particularly on the case $\psi = 0.8$. The sets of coefficients used in the factor $(1-\psi_1^* L - \psi_2^* L^2 - \psi_3^* L^3)$ in (33) are identical to those employed in the stationary component of (31) for computing size. The periodic cointegrating relationship in (33) has coefficients $\beta_4 = 0.4$, $\beta_{s-1} = 0.4\varphi^z_s/\varphi^y_s$ for $s = 3,2,1$, with $\varphi^y_s(s = 1,2,3,4)$ being the *PI* coefficients for $y_{s\tau}$. The innovation covariance matrix is again given by (32).

Table 1 shows the combinations of coefficients used in (30)–(33) to compute the empirical size and power. The size and power results are collected in Tables 2 and 3, respectively, for a sample size of 50 years (200 observations) and based on 5,000 replications.

TABLE 1. Periodic integration coefficients used for size and power calculations

DGP	φ^y_1	φ^y_2	φ^y_3	φ^y_4	φ^z_1	φ^z_2	φ^z_3	φ^z_4
1	1.200	0.700	1.000	1.190	0.800	0.900	1.200	1.157
2	1.200	1.000	0.800	1.042	0.800	1.000	1.200	1.042
3	0.800	0.800	1.200	1.302	1.200	0.700	1.000	1.190
4	1.200	0.700	1.000	1.190	1.200	0.700	1.000	1.190
5	0.800	1.000	1.200	1.042	0.800	1.000	1.200	1.042
6	0.800	0.900	1.200	1.157	0.800	0.900	1.200	1.157

Note: The table shows the periodic integration coefficients used for computing empirical size and power in Tables 1 and 2; see (30) and (33), respectively.

TABLE 2. Size of residual-based test for periodic cointegration

		$\psi_1 = \psi_2 = \psi_3 = 0$	$\psi_1 = 0.5, \psi_2 = \psi_3 = 0$		$\psi_1 = -0.3, \psi_2 = 0.4, \psi_3 = 0$			$\psi_1 = -0.8, \psi_2 = -0.6, \psi_3 = -0.4$			
σ_{yz}	DGP	PAR(1)	PAR(1)	PAR(2)	PAR(1)	PAR(2)	PAR(3)	PAR(1)	PAR(2)	PAR(3)	PAR(4)
0	1	0.047	0.006	0.050	0.097	0.003	0.045	0.857	0.420	0.141	0.031
0	2	0.040	0.003	0.041	0.104	0.004	0.046	0.887	0.474	0.176	0.033
0	3	0.046	0.005	0.045	0.103	0.003	0.045	0.862	0.452	0.167	0.036
0	4	0.044	0.005	0.051	0.099	0.003	0.049	0.855	0.427	0.153	0.040
0	5	0.049	0.005	0.052	0.102	0.005	0.045	0.887	0.490	0.188	0.043
0	6	0.036	0.006	0.051	0.110	0.004	0.052	0.881	0.464	0.173	0.044
0.4	1	0.041	0.006	0.051	0.109	0.003	0.048	0.846	0.398	0.133	0.033
0.4	2	0.042	0.006	0.047	0.094	0.005	0.041	0.878	0.464	0.164	0.033
0.4	3	0.047	0.006	0.041	0.106	0.002	0.047	0.856	0.437	0.167	0.042
0.4	4	0.048	0.005	0.048	0.107	0.003	0.048	0.852	0.415	0.140	0.041
0.4	5	0.042	0.006	0.053	0.104	0.004	0.051	0.892	0.492	0.180	0.034
0.4	6	0.049	0.006	0.047	0.107	0.004	0.051	0.871	0.471	0.174	0.043
0.8	1	0.042	0.005	0.049	0.095	0.003	0.041	0.819	0.364	0.108	0.029
0.8	2	0.043	0.005	0.049	0.110	0.005	0.048	0.842	0.423	0.150	0.028
0.8	3	0.038	0.004	0.045	0.101	0.003	0.044	0.833	0.399	0.143	0.032
0.8	4	0.040	0.005	0.049	0.113	0.003	0.050	0.851	0.413	0.140	0.039
0.8	5	0.047	0.005	0.048	0.107	0.004	0.048	0.891	0.499	0.187	0.042
0.8	6	0.045	0.007	0.051	0.114	0.005	0.048	0.880	0.464	0.174	0.041

Note: The residual-based test is applied to (7). Results are obtained using 5,000 replications, for a sample of 200 observations ($N = 50$). The DGP is given in (30)–(32), using the *PI* coefficients of Table 1 in combination with the (nonperiodic) stationary AR ψ_i coefficients common to both y and z given in the column headings. PAR(1), PAR(2), PAR(3), and PAR(4) indicate that periodic autoregressive models of order 1, 2, 3, or 4, respectively, are fitted to the residuals to obtain the LR statistic used to test periodic cointegration at a nominal significance level of 5%. The critical value used is 7.3, which has been obtained from a Monte Carlo analysis based on 15,000 replications of two uncorrelated $PI(1)$ processes with a sample size of 200 observations ($N = 50$).

Table 3. Power of residual-based test for periodic cointegration

		$\psi = 0$, $\psi_1^* = \psi_2^* = \psi_3^* = 0$	$\psi = 0.8$, $\psi_1^* = \psi_2^* = \psi_3^* = 0$	$\psi = 0.8$, $\psi_1^* = 0.5$, $\psi_2^* = \psi_3^* = 0$		$\psi = 0.8$, $\psi_1^* = -0.3$, $\psi_2^* = 0.4$, $\psi_3^* = 0$			$\psi = 0.8$, $\psi_1^* = -0.8$, $\psi_2^* = -0.6$, $\psi_3^* = -0.4$			
σ_{yz}	DGP	PAR(1)	PAR(1)	PAR(1)	PAR(2)	PAR(1)	PAR(2)	PAR(3)	PAR(1)	PAR(2)	PAR(3)	PAR(4)
0	1	1.000	0.986	0.324	0.997	0.986	0.781	0.997	1.000	1.000	1.000	0.991
0	2	1.000	0.980	0.309	0.997	0.972	0.786	0.996	1.000	1.000	1.000	0.993
0	3	1.000	0.985	0.301	0.997	0.991	0.756	0.997	1.000	1.000	1.000	0.992
0	4	1.000	0.987	0.306	0.998	0.987	0.771	0.995	1.000	1.000	1.000	0.992
0	5	1.000	0.983	0.299	0.998	0.986	0.749	0.997	1.000	1.000	1.000	0.993
0	6	1.000	0.980	0.297	0.999	0.987	0.765	0.996	1.000	1.000	1.000	0.993
0.4	1	1.000	0.978	0.319	0.998	0.975	0.758	0.996	1.000	1.000	1.000	0.991
0.4	2	1.000	0.969	0.323	0.996	0.969	0.784	0.997	1.000	1.000	1.000	0.989
0.4	3	1.000	0.982	0.295	0.998	0.985	0.738	0.997	1.000	1.000	1.000	0.987
0.4	4	1.000	0.976	0.295	0.997	0.982	0.746	0.997	1.000	1.000	1.000	0.987
0.4	5	1.000	0.978	0.286	0.998	0.988	0.739	0.996	1.000	1.000	1.000	0.987
0.4	6	1.000	0.978	0.297	0.998	0.985	0.741	0.996	1.000	1.000	1.000	0.989
0.8	1	1.000	0.966	0.283	0.998	0.973	0.703	0.993	1.000	1.000	1.000	0.986
0.8	2	1.000	0.960	0.297	0.997	0.955	0.728	0.994	1.000	1.000	0.999	0.983
0.8	3	1.000	0.978	0.271	0.998	0.984	0.704	0.992	1.000	1.000	1.000	0.987
0.8	4	1.000	0.972	0.283	0.997	0.976	0.692	0.994	1.000	1.000	1.000	0.980
0.8	5	1.000	0.967	0.268	0.998	0.979	0.684	0.995	1.000	1.000	1.000	0.984
0.8	6	1.000	0.973	0.260	0.999	0.972	0.693	0.994	1.000	1.000	1.000	0.981

Note: The residual-based test is applied to (7). Results are obtained using 5,000 replications, for a sample of 200 observations ($N = 50$). The DGP is given in (33), using the *PI* coefficients of Table 1 together with $\beta_4 = 0.4$, $\beta_{s-1} = 0.4\varphi_s^z/\varphi_s^y$ for $s = 3, 2, 1$, in combination with the (nonperiodic) stationary AR coefficients for y given in the column headings and with the innovation covariance defined in (32). PAR(1), PAR(2), PAR(3), and PAR(4) indicate that periodic autoregressive models of order 1, 2, 3, or 4, respectively, are fitted to the residuals to obtain the LR statistic used to test periodic cointegration at a nominal significance level of 5%. The critical value used is 7.3, which has been obtained from a Monte Carlo analysis based on 15,000 replications of two uncorrelated $PI(1)$ processes with a sample size of 200 observations ($N = 50$).

The results of Table 2 verify that, even in finite samples, the residual-based test for periodic cointegration has reasonably good size properties, provided that the appropriate order of PAR model is selected. In particular, the order p required in (20) is one greater than the order of the stationary AR component in (31). When a model of too low order is used, the test can be undersized or oversized, depending on the parameters of the process. In general, the satisfactory size (for appropriate p) applies across all sets of *PI* coefficients considered and irrespective of the extent of correlation between the disturbances. However, the final column suggests that, even for the correct p, the test tends to be undersized with increasing PAR order in the DGP.

Turning to the power results of Table 3, and again provided that an appropriate order of periodic process is fitted, the test has power approaching unity for all cases considered. It is unsurprising that unit power is obtained when the disturbances for $y_{s\tau}$ in (33) are white noise, compared to around 0.98 when these disturbances are a stationary AR(1) with coefficient 0.8. However, power does not diminish substantially when the process contains the root $(0.8)^{-1} = 1.25$ together with additional stationary autocorrelation. For example, with stationary AR(4) autocorrelation in a cointegrating *PI* regression, the test applied to a fitted PAR(4) process has power of at least 0.98 in Table 3. On the other hand, relatively low power is obtained when a PAR(1) is fitted in the context of the stationary AR(2) disturbance process, and to a lesser extent when a PAR(2) is fitted to an AR(3) disturbance DGP. This relatively low power is a consequence of estimating a model of too low order and reflects the undersizing evident in Table 2 for the corresponding cases, which have a (periodic) unit root in place of the stationary root of 1.25.

5.2. Deterministic Terms

To facilitate the preceding theoretical analysis, we omitted deterministic terms and assumed zero initial values, thereby implying $E[x_{s\tau}] = 0$. Here we relax these restrictions by considering the addition of deterministic terms to the cointegrating test regression.

In the case of standard (nonperiodic) cointegration, the appropriate form of the cointegration test regression depends on the properties of the time series under study; see Phillips and Ouliaris (1990) and Hansen (1992). The inclusion of an intercept allows for possibly nonzero starting values, with means constant over time, by demeaning the variables used in the long-run regression. The null distribution of the LR test for (nonperiodic) cointegration then satisfies (23), with $\overline{w_m}(r)$ as defined in (19), where it is understood that $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ is a vector of demeaned standard Brownian motions. The addition of a trend allows for a nonzero drift, and the vector of Brownian motions is then demeaned and detrended.

Turning to the case of *PI* processes, a nonzero starting value in (1) and no deterministic terms imply a seasonally varying mean $E[x_{s\tau}]$ that is constant over years $\tau = 1,2,\ldots$. However, as shown by Paap and Franses (1999), the addition of an intercept to (1) leads to a seasonally varying trend in $E[x_{s\tau}]$ and hence an annual growth rate $\Delta_4 x_{s\tau}$ that varies over $s = 1,2,3,4$. Further, excluding the special case of an $I(1)$ process, they show that a PI process with an intercept cannot have a trend that is common over $s = 1,2,3,4$, irrespective of whether the intercept is constant over seasons or is seasonally varying. On the other hand, and using a first-order process for ease of exposition, the process

$$x_{s\tau} = \mu_s + \vartheta_s \tau + \phi_s x_{s-1,\tau} + e_{s\tau}, \qquad \prod_{i=1}^{s} \phi_s = 1 \tag{34}$$

with $e_{s\tau}$ white noise and trend coefficients satisfying

$$\vartheta_s = (1-\phi_s)[\mu_4 + \phi_4 \mu_3 + \phi_3 \phi_4 \mu_2 + \phi_2 \phi_3 \phi_4 \mu_1], \qquad s = 1,2,3,4 \tag{35}$$

has a common linear trend shared by all quarters (Paap and Franses, 1999). However, with unrestricted trend coefficients, (34) implies seasonally varying quadratic trends in $E[x_{s\tau}]$.

In the context of testing for periodic cointegration, the preceding discussion implies that the relevant cointegrating regressions that may be considered in place of (7) are

$$x_{s\tau}^{(1)} = \beta_{0s} + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + v_{s\tau} \tag{36}$$

and

$$x_{s\tau}^{(1)} = \beta_{0s} + \beta_{1s}\tau + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + v_{s\tau}. \tag{37}$$

More specifically, (36) is appropriate when the variables in the regression are known to have constant (possibly periodically varying) mean over time, and the use of (37) permits the possibility that the variables may trend linearly over time.[8]

In addition to the case with unrestricted trends in (37), we also investigate cointegrating regressions using restricted trend coefficients such that $\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14}$. This last case is considered when ϑ_s satisfies the restrictions of (35) and hence the linear trend in each $PI(1)$ process is constant over seasons. All DGPs used in this analysis are uncorrelated PI PAR(1) processes.

The results of Table 4A verify that, for the three bivariate $PI(1)$ DGPs considered there, the inclusion of deterministic terms has the anticipated effect on the residual-based test for periodic cointegration. That is, for zero-mean processes, the inclusion of periodically varying intercepts or periodically varying intercepts and trends, as in (36) and (37), respectively, causes the distribution of the LR test for periodic cointegration to shift, with the percentiles of the test statistic under the null hypothesis being effectively the same as the correspond-

TABLE 4. Effect of deterministic terms on the empirical distribution of the residual-based periodic cointegration test

DGP	Deterministic terms in regression	Percentile						
		0.85	0.875	0.9	0.925	0.95	0.975	0.99
A. Zero-mean DGPs								
1	None	5.024	5.447	5.958	6.626	7.503	8.994	10.922
	Intercepts	8.062	8.565	9.154	9.943	10.981	12.709	14.970
	Intercepts and trends	10.824	11.386	12.113	12.992	14.149	16.047	18.484
2	None	5.001	5.402	5.904	6.572	7.438	8.908	10.748
	Intercepts	7.939	8.443	9.066	9.840	10.825	12.635	14.934
	Intercepts and trends	10.768	11.361	12.036	12.882	14.080	15.990	18.524
3	None	5.036	5.456	5.980	6.658	7.517	9.037	11.042
	Intercepts	8.034	8.553	9.158	9.884	10.949	12.767	15.014
	Intercepts and trends	10.840	11.382	12.067	12.910	14.110	16.026	18.492
B. Periodic-trend DGPs								
1	Intercepts and trends	10.844	11.421	12.108	12.948	14.158	16.062	18.505
2	Intercepts and trends	10.828	11.417	12.097	12.968	14.139	16.091	18.428
3	Intercepts and trends	10.789	11.389	12.073	12.930	14.089	16.023	18.488
C. Nonperiodic-trend DGPs								
1	Intercepts and trends	10.724	11.292	11.996	12.838	13.962	16.015	18.635
	Intercepts and restricted trend	26.281	28.984	32.485	37.367	44.460	56.823	75.590
2	Intercepts and trends	10.826	11.404	12.087	12.938	14.133	16.123	18.638
	Intercepts and restricted trend	16.689	18.021	19.562	21.827	25.219	31.502	40.791
3	Intercepts and trends	10.796	11.400	12.106	12.955	14.095	16.012	18.508
	Intercepts and restricted trend	17.053	18.498	20.272	22.663	26.285	32.921	43.194
D. Identical *PI* processes with nonperiodic trends								
1*	Intercepts and trends	10.854	11.419	12.187	13.014	14.132	16.061	18.761
	Intercepts and restricted trend	10.887	11.470	12.230	13.058	14.205	16.095	18.833
2*	Intercepts and trends	10.900	11.466	12.167	13.052	14.262	16.261	18.477
	Intercepts and restricted trend	10.937	11.505	12.223	13.110	14.285	16.351	18.613
3*	Intercepts and trends	10.787	11.440	12.163	13.051	14.230	16.020	18.523
	Intercepts and restricted trend	10.882	11.515	12.238	13.128	14.342	16.077	18.668
Phillips–Ouliaris critical values								
	None	5.100	5.538	6.005	6.668	7.628	9.331	11.468
	Intercept	8.202	8.744	9.399	10.228	11.326	13.264	15.696
	Intercept and trend	11.078	11.701	12.379	13.298	14.440	16.583	19.034

Note: The residual-based test is applied to (7) when no deterministic terms are included and to (36) or (37) as appropriate when intercepts or intercepts and trends are included in the regression. Intercepts and relevant trend coefficients are unrestricted, unless otherwise stated; restricted trends impose $\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14}$. All DGPs are uncorrelated (both serially and contemporaneously) *PI* bivariate PAR(1) processes. Using superscripts y and z to indicate the left- and right-hand-side variables, respectively, in (7), (36), or (37), the coefficients for the processes of A, B, and C are as follows: (1) $\phi_1^y = 0.8$, $\phi_2^y = 0.9$, $\phi_3^y = 1.2$, $\phi_4^y = 1.157$; $\phi_1^z = 1.2$, $\phi_2^z = 0.7$, $\phi_3^z = 1$, $\phi_4^z = 1.190$; (2) $\phi_1^y = 1.25$, $\phi_2^y = 0.8$, $\phi_3^y = 0.9$, $\phi_4^y = 1.111$; $\phi_1^z = 1$, $\phi_2^z = 0.8$, $\phi_3^z = 1.2$, $\phi_4^z = 1.042$; (3) $\phi_1^y = 1.2$, $\phi_2^y = 0.7$, $\phi_3^y = 1$, $\phi_4^y = 1.190$; $\phi_1^z = 0.8$, $\phi_2^z = 0.8$, $\phi_3^z = 1.2$, $\phi_4^z = 1.302$. The *PI* DGPs 1*, 2*, and 3* of D have periodic integration coefficients for both processes that are identical to the coefficients ϕ_s^y, $s = 1,2,3,4$, for DGPs 1, 2, and 3, respectively. The DGPs of B, C, and D use $\mu_1^y = 1$, $\mu_2^y = 1.2$, $\mu_3^y = 0.5$, $\mu_4^y = 0.2$; $\mu_1^z = 0.2$, $\mu_2^z = 0.5$, $\mu_3^z = 1.2$, $\mu_4^z = 1$ in a notation analogous to (34). These intercept values are also used in the nonperiodic-trend DGPs of C and D, with the trend coefficients restricted through (35). Results are based on 25,000 replications for a sample of size 2,000 observations ($N = 500$ years). The Phillips and Ouliaris (1990) percentiles are the squares of critical values given in their Tables IIa, IIb, and IIc corresponding to no deterministic terms, intercept, and intercept and trend, respectively, for $n = 1$ explanatory variable.

ing values obtained by Phillips and Ouliaris (1990) for the nonperiodic case (with the latter values squared).

Because the inclusion of unrestricted intercepts leads to seasonally varying trends in a $PI(1)$ process, a cointegrating test regression of the form of (37), with unrestricted intercepts and trends, takes account of these deterministic effects. Table 4B verifies that, in this case, the (squared) Phillips and Ouliaris (1990) critical values for nonperiodic random walks with drifts continue to apply in this periodic case. As seen in Table 4C, these critical values also apply if the individual processes within the DGP have trends restricted to be identical across seasons, provided that no restrictions are imposed when (37) is estimated. However, imposition of the restriction of nonperiodic trends in the cointegrating test regression of (37) causes the Phillips–Ouliaris critical values to be inappropriate for these DGPs.

In contrast to the effects of restricted trends in Table 4C, Table 4D shows that, whether the trend coefficients of the cointegrating test regression are restricted to be identical over seasons or not, the Phillips and Ouliaris (1990) critical values can be used when testing cointegration between two $PI(1)$ processes that have identical periodic coefficients, $\phi_s^{(j)} = \phi_s, j = y, z$. However, the case of identical coefficients across PI separate processes is a special one, for which Lemma 1 shows that any cointegration must be nonperiodic.

To investigate this further, consider the $PI(1)$ vector $x_{s\tau}$, where all elements have constant trends over seasons. Separating the deterministic and stochastic components of each element, we can write

$$x_{s\tau}^{(i)} = c_{0s}^{(i)} + c_1^{(i)}\tau + \xi_{s\tau}^{(i)}, \qquad s = 1,2,3,4; \qquad i = 1,2,\ldots,m, \tag{38}$$

where $\xi_{s\tau}^{(i)}$ is a zero-mean $PI(1)$ process and $E[x_{s\tau}^{(i)}] = c_{0s}^{(i)} + c_1^{(i)}\tau$, which has a periodically varying intercept but nonperiodic trend. The regression relevant for testing periodic cointegration between the zero-mean stochastic unit root processes $\xi_{s\tau}^{(i)}$ is

$$(x_{s\tau}^{(1)} - c_{0s}^{(1)} - c_1^{(1)}\tau) = \sum_{i=2}^{m} \beta_{is}(x_{s\tau}^{(i)} - c_{0s}^{(i)} - c_1^{(i)}\tau) + u_{s\tau},$$

that is,

$$\begin{aligned} x_{s\tau}^{(i)} &= \left(c_{0s}^{(1)} - \sum_{i=2}^{m} \beta_{is} c_{0s}^{(i)}\right) + \left(c_1^{(1)} - \sum_{i=2}^{m} \beta_{is} c_1^{(i)}\right)\tau + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + u_{s\tau} \\ &= \beta_{0s} + \beta_{1s}\tau + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + u_{s\tau}, \end{aligned} \tag{39}$$

which is identical in form to (37). Notice, however, that although (39) has periodically varying intercepts and periodic trend coefficients, the trend coefficients in the latter satisfy

$$\beta_{1s} = c_1^{(1)} - \sum_{i=2}^{m} \beta_{is} c_1^{(i)}, \qquad s = 1,2,3,4. \tag{40}$$

If the $PI(1)$ coefficients are identical across processes, and hence any cointegrating relationship is nonperiodic, then $\beta_{i1} = \beta_{i2} = \beta_{i3} = \beta_{i4}$ $(i = 2, \ldots, m)$, and (40) implies nonperiodic trends in the cointegrating regression of (37) or (39).

The Monte Carlo results of Table 4C and 4D support this analysis. In particular, the $PI(1)$ processes in Table 4D with identical coefficients and individual nonperiodic trends imply that any trend in (37) is also nonperiodic. Therefore, the restriction of identical trends derives from the nonperiodic nature of any cointegration in this case, with the imposition of this restriction effectively having no impact on the distribution of the residual-based test statistic.

On the other hand, when the $PI(1)$ coefficients differ over processes, (40) implies that the imposition of the nonperiodic trend restriction is inappropriate when the β_{is} are not correspondingly restricted. However, from Lemma 1, nonperiodic cointegration can apply only when the separate processes have identical *PI* coefficients. Therefore, the trend coefficients in (37) should not be restricted to be nonperiodic when testing for cointegration between periodic processes, except for the special case analyzed in Table 4D.

6. CONCLUDING REMARKS

This paper has provided an analysis of cointegration for periodically integrated processes. We first establish that the only cointegration possibilities are so-called full periodic or full nonperiodic cointegration. Because of the cointegration between seasons that exists for a univariate *PI* variable, if no cointegration between variables applies for a specific individual season, then no cointegration applies at all. Further, if the *PI* processes have identical coefficients over processes, then any cointegration that exists is nonperiodic, with identical cointegrating relationships over seasons.

This paper is the first to obtain analytical results for the distribution of two test statistics for cointegration, proposed as appropriate in previous literature for *PI* processes.

The available analytical results for cointegration related to seasonal processes have focused on the case of seasonally integrated processes, including Boswijk and Franses (1995), Hylleberg et al. (1990), and Johansen and Schaumburg (1999). However, the greater economic plausibility of periodic processes in some contexts suggests that attention should also be devoted to this case. The present paper provides results that contribute to our understanding of cointegration for seasonal processes, while also emphasizing that periodic and seasonal integration have distinct long-run implications. In particular, although the

Boswijk and Franses (1995) periodic cointegration test can be applied for both types of seasonal nonstationarity, the test statistic follows different distributions in the two cases. Therefore, a careful prior univariate analysis should be undertaken before considering cointegration for seasonal processes.

Our analysis also formally establishes the asymptotic distribution of a residual-based test of cointegration for *PI* processes, showing this distribution to be the same as for the nonperiodic case. Moreover, our Monte Carlo analysis verifies that the critical values of Phillips and Ouliaris (1990) can be used in the context of periodic processes, provided that potentially relevant trend terms included in the cointegration test regression are not restricted to be constant over the quarters of the year when the potential cointegration is periodic. Therefore, the test can be employed by applied workers in realistic contexts where the periodic series under analysis exhibit nonzero means and possible trends.

As in the case of univariate *PI* processes analyzed by Paap and Franses (1999), the use of trend terms in testing for periodic cointegration requires some care. Specifically, when testing for cointegration in periodic processes that contain nonperiodic trends, we show that the trend coefficients in the cointegration test regression should be restricted to be identical over seasons only when the individual processes have identical periodic coefficients. Because the situation where identical coefficients apply over the different univariate processes may not occur widely in practice, we recommend that the trend (and also the intercept) coefficients should be unrestricted over seasons when using the residual-based test for cointegration between *PI* processes.

NOTES

1. However, the starting conditions for the *SI* process may make this arbitrarily unlikely in finite samples.

2. Löf and Franses (2001) compare the forecast accuracy of periodic and seasonal cointegration models in a bivariate context.

3. Our focus is periodic cointegration, the analysis of which is facilitated by the assumption that the process in (1) has zero mean. However, we relax this assumption in the Monte Carlo analysis of Section 5.

4. Although periodic variation in Σ can be permitted, the purpose of our analysis is to examine the implications of periodically varying coefficients, and hence we assume constant disturbance covariances over seasons.

5. Note that, although (18) implies that the *PI* coefficients for the residuals are identical to those of the univariate process for $x_{s\tau}^{(1)}$ in (6), we do not propose that equality between these should be imposed.

6. See the paragraph between expressions (A.11) and (A.12) of Boswijk and Franses (1995, p. 451).

7. All results presented are based on the 5% critical value of 7.3 obtained through a Monte Carlo simulation for $T = 200$ observations. However, use of the asymptotic critical value of $(-2.76)^2$ from Phillips and Ouliaris (1990, Table IIa) does not alter the substantive conclusions.

8. In common with much of the unit root literature, the possibility of quadratic trends over time is excluded on a priori grounds.

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APPENDIX: Proofs

Proof of Lemma 1. To prove (i), and without loss of generality, assume that the linear combination $\alpha_1' x_{1\tau}$ is stationary, with α_1 of rank r. Also, for ease of exposition, assume two seasons per year τ, so that $s = 1, 2$.

The *PI* process of (6) then implies

$$x_{1\tau} = \Phi_1^+ x_{2,\tau-1} + U_{1\tau}, \tag{A.1}$$

where Φ_1^+ is a diagonal $m \times m$ matrix with jth diagonal element $\varphi_1^{(j)}$ and the stationary $m \times 1$ vector $U_{s\tau}$ has jth element $u_{s\tau}^{(j)}$. Premultiplying (A.1) by α_1' yields

$$\alpha_1' x_{1\tau} = \alpha_1' \Phi_1^+ x_{2,\tau-1} + \alpha_1' U_{1\tau} = \alpha_2' x_{2,\tau-1} + \alpha_1' U_{1\tau}, \tag{A.2}$$

where the $m \times r$ matrix $\alpha_2 = \Phi_1^+ \alpha_1$ defined by (A.2) has rank r, because Φ_1^+ is nonsingular and α_1 is of rank r. Further, the columns of α_2 must contain r cointegrating vectors for $x_{2,\tau-1}$, as otherwise the right-hand side of (A.2) would be nonstationary.

However, we need to establish that there are no additional linearly independent cointegrating vectors for $x_{2\tau}$, beyond those in the columns of α_2. Say one such cointegrating vector exists and append this as an additional column of α_2 to form the $m \times (r + 1)$ matrix α_2^* of rank $r + 1$. Then, analogously to (A.2), and where Φ_2^+ is a diagonal $m \times m$ matrix with jth diagonal element $\varphi_2^{(j)}$, we have

$$\alpha_2^{*\prime} x_{2\tau} = \alpha_2^{*\prime} \Phi_2^+ x_{1\tau} + \alpha_2^{*\prime} U_{2\tau} = \alpha_1^{*\prime} x_{1\tau} + \alpha_2^{*\prime} U_{2\tau}.$$

By the same argument as before, $\alpha_1^* = \Phi_2^+ \alpha_2^*$ must be a matrix of $r + 1$ cointegrating vectors for $x_{2\tau}$. This, however, contradicts the assumption that there are exactly r cointegrating vectors for $x_{1\tau}$. Consequently, there can be only r linearly independent cointegrating vectors for $x_{2\tau}$.

Recognizing that α_2 on the right-hand side of (A.2) relates to season $s - 1$ for $s = 1$, the generalization to four seasons, $s = 1, 2, 3, 4$, is straightforward, with the r cointegrating vectors for each quarter satisfying

$$\alpha_{s-1} = \Phi_s^+ \alpha_s, \qquad s = 1,2,3,4. \tag{A.3}$$

Note that for $s = 4$, it is understood that $s + 1 = 1$. By repeated substitution in (A.3), it is clear that given any α_s and the periodic coefficients, the cointegrating vectors for all other quarters can be determined. Also note that the *PI* property of (6) implies that

$$\Phi_1^+ \Phi_2^+ \Phi_3^+ \Phi_4^+ = I_4.$$

To establish (ii), first note that each of the m processes having identical *PI* coefficients implies $\Phi_s^+ = \varphi_s I_m$, for $s = 1,2,3,4$. Therefore, from (A.3), $\alpha_{s-1} = \varphi_s \alpha_s$, and because the scaling is irrelevant, the cointegrating relationships are identical over $s = 1$, 2, 3, 4. Conversely, because Φ_s^+ is nonsingular, $\alpha_s = c_s \alpha_{s-1}$ for some scalar constant c_s only if $\Phi_s^+ = c_s I_m$, $s = 1,2,3,4$. In turn, $\Phi_s^+ = c_s I_m$ implies that the m $PI(1)$ processes have identical periodic coefficients. ■

Proof of Lemma 2. Define the vector of observations for process j of $x_{s\tau}$ in year τ as $X_\tau^{(j)} = [x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)}]'$. As all elements of $x_{s\tau}$ are *SI*, then the series for the quarters of the year are not cointegrated, so that no $4 \times r$ matrix of cointegrating vectors β_j exists such that $\beta_j' X_\tau^{(j)}$ is stationary for any $j = 1, \dots, m$. Because no cointegration connects the integrated processes $x_{s\tau}$ and $x_{q\tau}$ $(q \neq s)$, the existence of cointegration between the elements of $x_{s\tau}$ has no implications for cointegration between the elements of $x_{q\tau}$. ■

Proof of Lemma 3. For the process of (6), (8), and (9), and as in Boswijk (1994) and Phillips and Ouliaris (1990), we use the decomposition $\Sigma = PP'$ where the upper triangular matrix P is

$$P = \begin{bmatrix} \sigma_{11}^{1/2}\sqrt{1-\rho_{1z}'\rho_{1z}} & \sigma_{11}^{1/2}\rho_{1z}' \\ 0 & P_{zz} \end{bmatrix}, \tag{A.4}$$

in which the $n \times 1$ (with $n = m - 1$) vector ρ_{1z} is defined from the elements of Σ in (9) as

$$\rho_{1z} = \sigma_{11}^{-1/2} P_{zz}^{-1} \sigma_{1z}. \tag{A.5}$$

For a $4m \times 1$ vector white noise sequence $\{E_\tau\}$ with mean zero and variance matrix I_{4m}, the multivariate invariance principle (see Phillips and Durlauf, 1986) implies that

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{[rN]} E_j \Rightarrow W(r), \tag{A.6}$$

where $W(r)$ is a $4m \times 1$ vector standard Brownian motion process. For later use, define

$$W(r) = [W^1(r)', W^2(r)', \dots, W^m(r)']' = [W^1(r)', W^z(r)']',$$

where $W^j(r)$, $j = 1, \dots, m$, are 4×1 vectors whose elements we associate with the seasons, while $W^z(r)$ is $4n \times 1$. From these, define the $4m \times 1$ vector Brownian motion with covariance matrix $\Sigma \otimes I_4$ as

$$E^x(r) = (P \otimes I_4) W(r) = \begin{bmatrix} \sigma_{11}^{1/2}\{\sqrt{1-\rho_{1z}'\rho_{1z}}\, W^{(1)}(r) + (\rho_{1z}' \otimes I_4) W^z(r)\} \\ (P_{zz} \otimes I_4) W^z(r) \end{bmatrix}. \tag{A.7}$$

As in Lemma 1 of Boswijk and Franses (1996),

$$\begin{aligned} \frac{1}{\sqrt{N}} X_{[rN]} &= (\Theta_0^x - \Theta_1^x) \frac{1}{\sqrt{N}} \sum_{j=1}^{[rN]} U_j^x + o_p(1) \\ &= (\Theta_0^x - \Theta_1^x) \frac{1}{\sqrt{N}} \Psi^x(1)^{-1} \sum_{j=1}^{[rN]} E_j^x + o_p(1) \\ &\Rightarrow B^x(r) = (\Theta_0^x - \Theta_1^x) \Psi^x(1)^{-1} E^x(r) = a^x b^{x\prime} \Psi^x(1)^{-1} E^x(r). \end{aligned} \tag{A.8}$$

Now, it is easy to see that

$$\Psi^x(1)^{-1}(P \otimes I_4) = \begin{bmatrix} \Psi^{(1)}(1)^{-1}\left(\sigma_{11}^{1/2}\sqrt{1-\rho_{1z}'\rho_{1z}}\right)I_4 & \Psi^{(1)}(1)^{-1}\sigma_{11}^{1/2}(\rho_{1z}' \otimes I_4) \\ 0 & \Psi^z(1)^{-1}(P_{zz} \otimes I_4) \end{bmatrix}.$$

Therefore, from (A.7) and (A.8), we have

$$\begin{aligned} B^{(1)}(r) &= \sigma_{11}^{1/2} a^{(1)} b^{(1)\prime} \Psi^{(1)}(1)^{-1}\left(\sqrt{1-\rho_{1z}'\rho_{1z}}W^{(1)}(r) + (\rho_{1z}' \otimes I_4)W^z(r)\right), \\ B^z(r) &= a^z b^{z\prime} \Psi^z(1)^{-1}(P_{zz} \otimes I_4)W^z(r), \end{aligned} \tag{A.9}$$

as in (15) of the text, where a^z and b^z are defined as the lower right-hand $4n \times n$ blocks of a^x and b^x in (14). ■

Proof of Lemma 4. Consider first the OLS estimates of the coefficients of (7) for each season, denoted $\hat{B}_s = [\hat{\beta}_{2s}, \hat{\beta}_{3s}, \ldots, \hat{\beta}_{ms}]'$, where

$$\hat{B}_s = \left[N^{-2}\sum_{\tau=1}^{N} z_{s\tau} z_{s\tau}'\right]^{-1}\left[N^{-2}\sum_{\tau=1}^{N} x_{s\tau}^{(1)} z_{s\tau}\right].$$

Then

$$\hat{B}_s \Rightarrow \left[\int B_s^z(r)B_s^z(r)'\,dr\right]^{-1}\int B_s^z(r)B_s^{(1)}(r)\,dr, \tag{A.10}$$

where $B_s(r) = [B_s^{(1)}(r), B_s^z(r)']'$ is $m \times 1$ vector Brownian motion, with $n \times 1$ $B_s^z(r) = [B_s^{(2)}(r), \ldots, B_s^{(m)}(r)]'$.

From (17) and defining the $n \times 1$ vector $\widetilde{W}^z(r) = [\widetilde{w}^{(2)}(r), \widetilde{w}^{(3)}(r), \ldots, \widetilde{w}^{(m)}(r)]'$, we have

$$\begin{aligned} \int B_s^z(r)B_s^z(r)'\,dr &= A_s\left[\int \widetilde{W}^z(r)\widetilde{W}^z(r)'\,dr\right]A_s, \\ \int B_s^z(r)B_s^{(1)}(r)\,dr &= A_s\left[\int \widetilde{W}^z(r)\widetilde{w}^{(1)}(r)\,dr\right]\omega_1 a_s^{(1)}, \end{aligned} \tag{A.11}$$

where A_s is an $n \times n$ diagonal matrix such that $A_s = diag\{w_2 a_s^{(2)}, \omega_3 a_s^{(3)}, \ldots, \omega_m a_s^{(m)}\}$.

Then, from (A.10) and (A.11) it is easy to see that

$$\hat{B}_s \Rightarrow \omega_1 a_s^{(1)} A_s^{-1}[\widetilde{W}^z(r)\widetilde{W}^z(r)']^{-1}\int \widetilde{W}^z(r)\widetilde{w}^{(1)}(r)\,dr. \tag{A.12}$$

The appropriately scaled residuals from (7) can be expressed as

$$\frac{1}{\sqrt{N}}\hat{v}_{s[rN]} = \frac{1}{\sqrt{N}}x_{s[rN]}^{(1)} - \hat{B}_s'\frac{1}{\sqrt{N}}z_{s[rN]}, \tag{A.13}$$

where $z_{s[rN]} = [x^{(2)}_{s[rN]}, x^{(3)}_{s[rN]}, \ldots, x^{(m)}_{s[rN]}]'$ and, from (17),

$$\frac{1}{\sqrt{N}} z_{s[rN]} \Rightarrow B^z_s(r) = A_s \widetilde{W}^z(r),$$
$$\frac{1}{\sqrt{N}} x^{(1)}_{s[rN]} \Rightarrow B^{(1)}_s(r) = \omega_1 a^{(1)}_s \widetilde{w}^{(1)}(r). \tag{A.14}$$

Hence, from (A.12)–(A.14),

$$\frac{1}{\sqrt{N}} \hat{v}_{s[rN]} \Rightarrow \omega_1 a^{(1)}_s \left\{ \widetilde{w}^{(1)}(r) - \int \widetilde{w}^{(1)}(r)\widetilde{W}^z(r)'\,dr \left[\int \widetilde{W}^z(r)\widetilde{W}^z(r)'\,dr \right]^{-1} \widetilde{W}^z(r) \right\}$$
$$= \omega_1 a^{(1)}_s \eta' \widetilde{W}^x(r), \tag{A.15}$$

where $\eta' = [1, -\int \widetilde{w}^{(1)}(r)\widetilde{W}^z(r)'\,dr[\int \widetilde{W}^z(r)\widetilde{W}^z(r)'\,dr]^{-1}]$. Each element of the $m \times 1$ vector of Brownian motions $\widetilde{W}^x(r) = [\widetilde{w}^{(1)}(r), \widetilde{W}^z(r)']'$ has unit variance, and the $m \times m$ long-run covariance matrix Π of $\widetilde{W}^x(r)$ can be expressed as

$$\Pi = \begin{bmatrix} 1 & \boldsymbol{\varpi}'_{1z} \\ \boldsymbol{\varpi}_{1z} & \Pi_{zz} \end{bmatrix}$$

with elements on the principal diagonal of Π_{zz} equal to one.

Defining the $m \times m$ matrix L such that $\Pi = LL'$, and where the first column of L is given by $(l_{11}, 0_n)$, then, using part (a) of Lemma 2.2 of Phillips and Ouliaris (1990), we have that

$$\widetilde{W}^x(r) = L\overline{W^x}(r),$$

where $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{w^{(2)}}(r), \ldots, \overline{w^{(m)}}(r)]' = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ is an $m \times 1$ vector of standard Brownian motions with covariance matrix I_m. Finally from part (b) of Lemma 2.2 of Phillips and Ouliaris (1990), it is possible to write

$$\eta' \widetilde{W}^x(r) = l_{11} k' \overline{W^x}(r),$$
$$k' = \left[1, -\int \overline{w^{(1)}}(r)\overline{W^z}(r)'\,dr \left(\int \overline{W^z}(r)\overline{W^z}(r)'\,dr \right)^{-1} \right].$$

Recalling that $\omega_1 = \sigma^{0.5}_{11}(b^{(1)\prime}\Psi^{(1)}(1)^{-1}\Psi^{(1)}(1)'^{-1}b^{(1)})^{0.5}$, the result in (18)–(19) is obtained by substituting these last two expressions into (A.15) and stacking the residuals for $s = 1,2,3,4$ to define the vector $\hat{V}_{[rN]} = [\hat{v}_{1[rN]}, \hat{v}_{2[rN]}, \hat{v}_{3[rN]}, \hat{v}_{4[rN]}]'$. ■

Proof of Theorem 1. It follows from Lemma 4 that, in the absence of cointegration, we can write

$$\hat{v}_{s\tau} = \varphi^{(1)}_s \hat{v}_{s-1,\tau} + u^m_{s\tau}, \qquad \prod_{s=1}^{4} \varphi^{(1)}_s = 1. \tag{A.16}$$

Now define $U_\tau^m = [u_{1\tau}^m, u_{2\tau}^m, u_{3\tau}^m, u_{4\tau}^m]' = \Psi^{(1)}(L^4)^{-1} E_\tau^m$ and $E_\tau^m = [\varepsilon_{1\tau}^m, \varepsilon_{2\tau}^m, \varepsilon_{3\tau}^m, \varepsilon_{4\tau}^m]'$. As in Lemma 1 of Boswijk and Franses (1996) and Lemma 3 of the text, we have that

$$\frac{1}{\sqrt{N}} \hat{V}_{[rN]} \Rightarrow \sigma(\Theta_0^{(1)} - \Theta_1^{(1)}) U^m(r) = \sigma a^{(1)} b^{(1)\prime} \Psi^{(1)}(1)^{-1} E^m(r) \tag{A.17}$$

with $1/\sqrt{N}\sum_{j=1}^{[rN]} E_j^m \Rightarrow \sigma E^m(r)$ and $E^m(r)$ is 4×1 vector standard Brownian motion that is a function of the elements of $W(r)$. Comparing (18) with (A.17), it follows that

$$\sigma a^{(1)} b^{(1)\prime} \Psi^{(1)}(1)^{-1} E^m(r) = \sigma_{11}^{1/2} l_{11} \omega a^{(1)} \overline{w_m}(r),$$

where $\omega = (b^{(1)\prime} \Psi^{(1)}(1)^{-1} \Psi^{(1)}(1)^{\prime -1} b^{(1)})^{0.5}$, and hence

$$\overline{w_m}(r) = \frac{\sigma}{l_{11}\sigma_{11}^{1/2}\omega} b^{(1)\prime} \Psi^{(1)}(1)^{-1} E^m(r) \tag{A.18}$$

provides an alternative definition of $\overline{w_m}(r)$. Also note that from Lemma 4 we know that $\overline{w_m}(r) = k'\overline{W^x}(r)$, and hence it is easy to see that the variance of $\overline{w_m}(r)$ is equal to $k'k$. Hence, from (A.18), $\sigma = l_{11}\sigma_{11}^{1/2}(k'k)^{1/2}$. For notational convenience in what follows, we omit the superscripts when referring to the periodic integration coefficients relating to process $x_{s\tau}^{(1)}$ and hence simply refer to a and b.

The theorem can then be established in a similar way to Theorem 1 in Boswijk and Franses (1996). It is convenient to write (20)–(21) using conventional time subscripts and seasonal dummy variable notation (D_{st} taking the value unity when observation t falls in season s and zero otherwise). Employing this notation yields the following representation (see Boswijk and Franses, 1996, p. 238):

$$\hat{v}_t = \pi_1 D_{1t}\hat{v}_{t-1} + \sum_{s=1}^{4} \varphi_s D_{st}\hat{v}_{t-1} + \sum_{s=1}^{4}\sum_{j=1}^{p-1} \psi_{js}(D_{st}\hat{v}_{t-j} - \varphi_{s-j} D_{st}\hat{v}_{t-j-1}) + \varepsilon_t, \tag{A.19}$$

where the restriction $\varphi_1\varphi_2\varphi_3\varphi_4 = 1$ is imposed. Let $\theta = [\theta_1, \theta_2', \theta_3']'$ denote the full parameter vector with $\theta_1 = \pi_1$, $\theta_2' = [\varphi_2, \varphi_3, \varphi_4]$, and $\theta_3' = [\psi_{11}, \ldots, \psi_{1,p-1}, \ldots, \psi_{41}, \ldots, \psi_{4,p-1}]$. Under the null hypothesis $\pi_1 = 0$, this parameter is associated with the unit root while φ_2, φ_3, and φ_4 are cointegration parameters (with φ_1 defined from the periodic unit root restriction as $\varphi_1 = (\varphi_2\varphi_3\varphi_4)^{-1}$), and θ_3 collects the parameters associated with the stationary regressors in (A.19).

Let $z_t = [z_t^1, z_t^{2\prime}, z_t^{3\prime}]'$ be defined conformably with θ as $z_t = \partial\hat{v}_t/\partial\theta$, and hence

$$z_t^1 = D_{1t}\hat{v}_{t-1}, \qquad z_t^2 = H'\hat{u}_t, \qquad \hat{u}_t = [\hat{u}_{1t}, \hat{u}_{4t}, \hat{u}_{3t}, \hat{u}_{4t}]',$$

where

$$\hat{u}_{st} = D_{st}\hat{v}_{t-1} - \sum_{i=1}^{p-1} \psi_{i,s+i} D_{s+i,t}\hat{v}_{t-i-1}, \qquad s = 1,2,3,4$$

and

$$H' = \begin{bmatrix} -\frac{\varphi_1}{\varphi_2} & 1 & 0 & 0 \\ -\frac{\varphi_1}{\varphi_3} & 0 & 1 & 0 \\ \frac{\varphi_1}{\varphi_4} & 0 & 0 & 1 \end{bmatrix}.$$

From Lemma 3 and (A.17), z_t^1 and z_t^2 satisfy

$$\begin{aligned}
&N^{-1}\sum_{t=1}^{T} z_t^1 \varepsilon_t \Rightarrow l_{11}\sigma_{11}^{1/2}\omega\sigma a_4 \int \overline{w_m}(r)\, dE_1^m(r),\\
&N^{-2}\sum_{t=1}^{T} (z_t^1)^2 \Rightarrow l_{11}^2\sigma_{11}\omega^2 a_4^2 \int \overline{w_m}(r)^2\, dr,\\
&N^{-2}\sum_{t=1}^{T} z_t^2 z_t^{2\prime} \Rightarrow l_{11}^2\sigma_{11}\omega^2 H' A^* \Psi^{(1)}(1)'\Psi^{(1)}(1) A^* H \int \overline{w_m}(r)^2\, dr,\\
&N^{-2}\sum_{t=1}^{T} z_1^2 z_t^1 \Rightarrow l_{11}^2\sigma_{11}\omega^2 H' A^* \Psi^{(1)}(1)' A_1^* \int \overline{w_m}(r)^2\, dr,\\
&N^{-1}\sum_{t=1}^{T} z_t^2 \varepsilon_t \Rightarrow l_{11}\sigma_{11}^{1/2}\omega\sigma H' A^* \Psi^{(1)}(1)' \int \overline{w_m}(r)\, dE^m(r),
\end{aligned} \quad \textbf{(A.20)}$$

where $\omega = (b'\Psi^{(1)}(1)^{-1}\Psi^{(1)}(1)'^{-1}b)^{0.5}$, $A^* = diag(a_4, a_1, a_2, a_3) = diag(\varphi_2\varphi_3\varphi_4,\ 1,\ \varphi_2, \varphi_2\varphi_3)$, $A_1^* = (a_4,\ 0,\ 0,\ 0)'$. Note that in our case the processes $E_s^m(r)$ for $s = 1, 2, 3, 4$ have unit variance.

Under the periodic unit root null hypothesis, the PAR$(p-1)$ regressors $D_{st}\hat{v}_{t-j} - \varphi_s D_{st}\hat{v}_{t-j-1}$ collected in the vector z_t^3 are stationary, with

$$\begin{aligned}
&N^{-1/2}\sigma^{-2}\sum_{t=1}^{T} z_t^3 \varepsilon_t \Rightarrow N(0, V_3),\\
&N^{-1}\sigma^{-2}\sum_{t=1}^{T} z_t^3 z_t^{3\prime} \to V_3.
\end{aligned} \quad \textbf{(A.21)}$$

Finally, reflecting the different rates of convergence for the parameter estimates corresponding to the nonstationary *PI* regressors and those for the stationary PAR$(p-1)$ component in the augmented regression (20) or (21), we have that

$$\begin{aligned}
&N^{-1}\sum_{t=1}^{T} z_t^3 z_t^{2\prime} = O_p(1),\\
&N^{-1}\sum_{t=1}^{T} z_t^3 z_t^1 = O_p(1).
\end{aligned}$$

The distribution of the LR test is established by Boswijk and Franses (1996) using

$$LR = \frac{(N\hat{\theta}_1)^2}{(Y_N^{-1} Q_\theta Y_N^{-1})^{11}} + o_p(1), \quad \textbf{(A.22)}$$

where $Y_N = diag(N \times I_4, N^{1/2} \times I_{4(p-1)})$, $(Y_N^{-1} Q_\theta Y_N^{-1})^{11}$ is the first element of the principal diagonal of the inverse matrix $(Y_N^{-1} Q_\theta Y_N^{-1})^{-1}$, $N\hat{\theta}_1$ is the first element of $(Y_N^{-1} Q_\theta Y_N^{-1})^{-1} Y_N^{-1} q_\theta$, and q_θ and Q_θ are the score vector and negative of the Hessian matrix, respectively, formulated in terms of θ.

Note that, as in Boswijk and Franses (1996),

$$(Y_N^{-1} Q_\theta Y_N^{-1})^{-1} Y_N^{-1} q_\theta = \left(\frac{1}{\sigma^2} Y_N^{-1} \sum z_t z_t' Y_N^{-1} \right)^{-1} \frac{1}{\sigma^2} Y_N^{-1} \sum z_t \varepsilon_t$$

and hence, from (A.20) and (A.21), it is easy to see that

$$Y_N^{-1} Q_\theta Y_N^{-1} \Rightarrow \begin{bmatrix} \dfrac{l_{11}^2 \sigma_{11} \omega^2}{\sigma^2} K'K \displaystyle\int \overline{w_m}(r)^2\, dr & 0 \\ 0 & V_3 \end{bmatrix},$$

$$Y_N^{-1} q_\theta \Rightarrow \begin{bmatrix} \dfrac{l_{11} \sigma_{11}^{1/2} \omega}{\sigma} K' \displaystyle\int \overline{w_m}(r)\, dE^m(r) \\ N(0, V_3) \end{bmatrix}, \quad \textbf{(A.23)}$$

where we define $K = [A_1^* \vdots \Psi^{(1)}(1) A^* H]$. Therefore,

$$(Y_N^{-1} Q_\theta Y_N^{-1})^{-1} Y_N^{-1} q_\theta \Rightarrow \begin{bmatrix} \dfrac{\sigma}{l_{11} \sigma_{11}^{1/2} \omega} \left\{ \displaystyle\int \overline{w_m}(r)^2\, dr \right\}^{-1} \displaystyle\int \overline{w_m}(r)\, d(K'K)^{-1} K' E^m(r) \\ N(0, V_3^{-1}) \end{bmatrix}.$$

(A.24)

Now, partitioning $K = [K_1 \vdots K_2]$ to focus on the first element of $(Y_N^{-1} Q_\theta Y_N^{-1})^{-1} Y_N^{-1} q_\theta$, namely, $N\hat{\theta}_1$, then (A.24) implies

$$N\hat{\theta}_1 \Rightarrow \left\{ \int w_m(r)^2\, dr \right\}^{-1} \int \overline{w_m}(r)\, dS_1(r), \quad \textbf{(A.25)}$$

where

$$S_1(r) = \frac{\sigma}{l_{11} \sigma_{11}^{1/2} \omega} (K_1' M_2 K_1)^{-1} K_1' M_2 E^m(r) \quad \textbf{(A.26)}$$

and $M_2 = I - K_2 (K_2' K_2)^{-1} K_2'$.

As in Boswijk and Franses (1996), the covariances between $S_1(r)$ and $K_2' E^m(r)$ and between $\overline{w_m}(r)$ and $K_2' E^m(r)$ are easily seen to be zero. Thus both $S_1(r)$ and $\overline{w_m}(r)$ are independent of the 3×1 vector Brownian motion $K_2' E^m(r)$, and, because these are all defined from the same 4×1 vector Brownian motion $E^m(r)$, it follows that $S_1(r)$ and $\overline{w_m}(r)$ must be the same up to a scale factor. From (A.26), noting that $K_1' M_2 K_1$ is scalar, the variance of $S_1(r)$ is seen to be $\sigma^2 l_{11}^{-2} \sigma_{11}^{-1} \omega^{-2} (K_1' M_2 K_1)^{-1}$, and (A.18) implies that the variance of $\overline{w_m}(r)$ is given by $\sigma^2 l_{11}^{-2} \sigma_{11}^{-1}$. Consequently,

$$S_1(r) = \omega^{-1} (K_1' M_2 K_1)^{-1/2} \overline{w_m}(r). \tag{A.27}$$

Substituting for $S_1(r)$ from (A.27) into (A.25) yields

$$N\hat{\theta}_1 \Rightarrow \omega^{-1} (K_1' M_2 K_1)^{-1/2} \left\{ \int w_m(r)^2 \, dr \right\}^{-1} \int \overline{w_m}(r) \, d\overline{w_m}(r). \tag{A.28}$$

From (A.23),

$$\begin{aligned} (Y_N^{-1} Q_\theta Y_N^{-1})^{11} &\Rightarrow \frac{\sigma^2}{l_{11}^2 \sigma_{11} \omega^2} (K'K)^{11} \left(\int \overline{w_m}(r)^2 \, dr \right)^{-1} \\ &= \frac{\sigma^2}{l_{11}^2 \sigma_{11} \omega^2} (K_1' M_2 K_1)^{-1} \left(\int \overline{w_m}(r)^2 \, dr \right)^{-1}. \end{aligned} \tag{A.29}$$

The result in (23) is easily obtained by substituting (A.28) and (A.29) into (A.22) and using $\sigma = l11\, \sigma 11\; 1/2 (k\prime k) 1/2$. ■

Proof of Theorem 2. For (i), note, first, that the Wald statistic (25) to test the null for no cointegration in season s is

$$\begin{aligned} Wald_s &= \hat{\delta}' (\hat{\mathrm{Var}}[\delta_s]^{-1}) \hat{\delta}_s \\ &= \hat{\sigma}_u^{-2} \left(\sum_{\tau=1}^{N} \Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1}' \right) \left(\sum_{\tau=1}^{N} x_{s,\tau-1} x_{s,\tau-1}' \right)^{-1} \left(\sum_{\tau=1}^{N} \Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1} \right). \end{aligned} \tag{A.30}$$

Then, from Lemma 3,

$$N^{-1} \sum_{\tau=1}^{N} \Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1} = \int B_s^x(r) \, dB_s^{(1)}(r),$$

$$N^{-2} \sum_{\tau=1}^{N} x_{s,\tau-1}' x_{s,\tau-1} \Rightarrow \int B_s^x(r) B_s^x(r)' \, dr,$$

where $B_s^x(r) = [B_s^{(1)}, B_s^{(2)}, \ldots, B_s^{(m)}]'$. Using (17) with $\Psi^{(1)}(1) = 1$, and the fact that in the spurious regression case $\Sigma = diag\{\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}\}$, it is possible to see that

$$B_s^{(j)}(r) = \sigma_{jj}^{1/2} a_s^{(j)} (b^{(j)\prime} b^{(j)})^{1/2} \overline{w^{(j)}}(r), \qquad j = 1, 2, \ldots, m.$$

Substituting into (A.30) it then follows that

$$
\begin{aligned}
Wald_s &= \hat{\delta}_s'(\hat{\text{Var}}[\delta_s]^{-1})\hat{\delta}_s \\
&\Rightarrow \sigma_u^{-2}\left(\int B_s^x(r)\,dB_s^{(1)}(r)\right)'\left(\int B_s^x(r)B_s^x(r)'\,dr\right)^{-1}\left(\int B_s^x(r)\,dB_s^{(1)}(r)\right) \\
&= \sigma_u^{-2}\sigma_{11}(a_s^{(1)})^2 b^{(1)\prime}b^{(1)}\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'\,dr\right)^{-1} \\
&\quad \times \left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right), \qquad \textbf{(A.31)}
\end{aligned}
$$

where, as in Boswijk and Franses (1996),

$$
\hat{\sigma}_u^2 \to \sigma_u^2 = \frac{1}{4}\sum_{s=1}^{4}\text{Var}(x_{s\tau}^{(1)}) = \frac{1}{4}\sigma_{11}b^{(1)\prime}b^{(1)}a^{(1)\prime}a^{(1)}. \qquad \textbf{(A.32)}
$$

Substituting (A.32) in (A.31) yields the result in (27).

For the joint test statistic, because of the seasonal dummy variables,

$$
\begin{aligned}
Wald &= \sum_{s=1}^{4} Wald_s \\
&= 4\,\frac{\sum_{s=1}^{4}(a_s^{(1)})^2}{a^{(1)\prime}a^{(1)}}\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'\,dr\right)^{-1}\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right) \\
&= 4\left(\int \overline{W^x}(r)\,d\overline{w^x}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'\,dr\right)^{-1}\left(\int \overline{W^x}(r)\,d\overline{w^{(1)}}(r)\right)
\end{aligned}
$$

as given in (28). ■