Journal of Applied Statistics, Vol. 27, No. 8, 2000, 1045–1049

Choosing a lower specification limit for an exponential process with 'the larger the better' tolerance: a simple, exact solution

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ABSTRACT *Chen (1999) proposed an economic design, using Taguchi's quality loss function, for choosing a producer's lower specification limit η for a product with a quality characteristic that has an exponential distribution with mean θ and 'the larger the better' tolerance. Chen (1999) developed an approximate solution that is applicable when $0.5 \leqslant \eta/\theta \leqslant 0.7$ and that requires numerical minimization. We derive a simple, exact solution that is applicable for all values of η/θ and does not require numerical minimization.*

1 Introduction

Taguchi (1986) and Taguchi *et al.* (1989) suggested using a quadratic quality loss function as the basis of economic design of product manufacturing tolerances. This quadratic loss function penalized deviations of the product's quality characteristic from its target, enabling a trade-off between the cost of tight tolerance and product quality. When the dimension of the quality characteristic is 'the larger the better', Taguchi recommended choosing the producer's lower specification limit using the *expected quality loss per item*, $E(L)$, defined as

$$E(L) = A_0 \Delta_0^2 E\left(\frac{1}{X^2}\right) \tag{1}$$

where A_0 is the loss caused by the product performing unsatisfactorily, Δ_0 is the dimension below which the product performs unsatisfactorily, and X is the dimension of the quality characteristic of the product.

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ISSN 0266-4763 print; 1360-0532 online/00/081045-05

Kapur & Wang (1987) developed an economic design, using Taguchi's quality loss function (1), for choosing a specification limit(s) for a product's quality characteristic. They suggested that the producer choose a specification limit(s) so as to minimize the expected *total* loss per item represented by

$$E(L_T) = E(L) \times \text{[acceptable fraction of output (i.e. shipped output)]} \\ + C_s \times \text{[unacceptable fraction of output (i.e. scrapped output)]} \\ + C_i \qquad (2)$$

where C_s is the scrap cost per item and C_i is the inspection cost per item.

To deal with the case where the quality characteristic had 'the larger the better' tolerance, Kapur & Wang (1987) used a truncated Taylor series expansion of $1/x^2$ around $E(X)$ to derive an approximation to $E(1/X^2)$ that, when substituted in (1), provided an approximation to $E(L)$ for use in (2). Kapur & Wang (1987) then used this approximation to $E(L)$ to derive an approximate form for $E(L_T)$ when the quality characteristic was normally distributed.

2 Chen's approximation

Chen (1999) addressed the case were the quality characteristic had 'the larger the better' tolerance and the quality characteristic was exponentially distributed.

If the quality characteristic of a product has an exponential distribution with mean θ but the producer chooses a lower specification limit η, where $\eta > \Delta_0$, then the probability density function of the quality characteristic Y of the products shipped to the customer will be

$$f(y) = \frac{e^{-y/\theta}}{\theta e^{-\eta/\theta}}, \qquad 0 < \eta < \theta < \infty, 0 < \eta < y < \infty \qquad (3)$$

For this distribution, Chen (1999), using the approximation to $E(L)$ of Kapur & Wang (1987), derived

$$E(L) = A_0 \Delta_0^2 E\left(\frac{1}{Y^2}\right) \approx A_0 \Delta_0^2 \frac{1 + 3\theta^2/(\theta+\eta)^2}{(\theta+\eta)^2}$$

which was shown to be accurate when $0.5 \leqslant \eta/\theta \leqslant 0.7$.

Following the recommendation of Kapur & Wang (1987), Chen (1999) then suggested choosing the lower specification limit η for exponential quality characteristics so as to minimize the following approximate expected total loss per item,

$$\begin{aligned} E(L_T) &= E(L)\, e^{-\eta/\theta} + C_s(1 - e^{-\eta/\theta}) + C_i \\ &\approx A_0 \Delta_0^2 \frac{1 + 3\theta^2/(\theta+\eta)^2}{(\theta+\eta)^2} e^{-\eta/\theta} + C_s(1 - e^{-\eta/\theta}) + C_i \end{aligned} \qquad (4)$$

as long as the value of η that minimizes equation (4) falls in the range $0.5 \leqslant \eta/\theta \leqslant 0.7$.

Let $\hat{\eta}_{\min}$ be the estimated value of η found by minimizing the above approximate expression (4) for $E(L_T)$ and let $\eta_{\min}$ be value of η that minimizes the true $E(L_T)$.

Chen emphasized that the minimization of $E(L_T)$ in (4) had to be carried out by numerical methods, and that $\hat{\eta}_{\min}$ could only approximate $\eta_{\min}$ if

(a) $0.5 \leqslant \hat{\eta}_{\min}/\theta \leqslant 0.7$ (i.e. $\hat{\eta}_{\min}$ is in the range of the approximation), and
(b) $0.5 \leqslant \eta_{\min}/\theta \leqslant 0.7$ (i.e. $\eta_{\min}$ is in the range of the approximation).

We now derive an exact expression for the expected quality loss per item, $E(L)$, and then show that the minimum of the resulting exact expected total loss per item is simply

$$\eta_{\min} = \sqrt{A_0 \Delta_0^2 / C_s}$$

3 An exact expression for *E(L)*

For a product whose critical dimension is a truncated exponential random variable Y with probability density function (3), and where A_0 is the loss caused by the product performing unsatisfactorily and Δ_0 is the dimension below which the product performs unsatisfactorily, the expected quality loss per item is

$$E(L) = A_0 \Delta_0^2 E\left(\frac{1}{Y^2}\right) = \frac{A_0 \Delta_0^2}{\theta \, \mathrm{e}^{-\eta/\theta}} \int_\eta^\infty \frac{\mathrm{e}^{-y/\theta}}{y^2} \mathrm{d}y$$

Integrating by parts and then using the transformation $t = y/\theta$ gives

$$E(L) = \frac{A_0 \Delta_0^2}{\theta \, \mathrm{e}^{-\eta/\theta}} \left[\frac{\mathrm{e}^{-\eta/\theta}}{\eta} - \frac{1}{\theta} \int_\eta^\infty \frac{\mathrm{e}^{-y/\theta}}{y} \mathrm{d}y \right]$$

$$= \frac{A_0 \Delta_0^2}{\theta \mathrm{e}^{-\eta/\theta}} \left[\frac{\mathrm{e}^{-\eta/\theta}}{\eta} - \frac{1}{\theta} \Gamma(0, \eta/\theta) \right]$$

where $\Gamma(a, x)$ is the incomplete gamma function defined by $\Gamma(a, x) = \int_x^\infty \mathrm{e}^{-t} t^{a-1} \mathrm{d}t$ (Abramowitz & Stegun, 1970, eqn (6.5.3)).

This simplifies to

$$E(L) = A_0 \Delta_0^2 \left[\frac{1}{\eta\theta} - \frac{\Gamma(0, \eta/\theta)}{\theta^2 \mathrm{e}^{-\eta/\theta}} \right] \tag{5}$$

4 The minimum of *E(L$_T$)*

Using the exact expected quality loss per item (5) in the Kapur & Wang (1987) formulation (2) gives the following exact expected total loss per item:

$$E(L_T) = \frac{A_0 \Delta_0^2}{\theta^2} \left[\frac{\mathrm{e}^{-\eta/\theta}}{\eta/\theta} - \Gamma(0, \eta/\theta) \right] + C_s(1 - \mathrm{e}^{-\eta/\theta}) + C_i \tag{6}$$

We now show that $\eta_{\min}$, the value of η that minimizes the exact expected total loss per item given in equation (6), can be expressed in a simple, closed form.

Since $\partial/\partial x \Gamma(a, x) = -x^{a-1} \mathrm{e}^{-x}$ (Abramowitz & Stegun, 1970, eqn (6.5.25)), we find

$$\frac{\partial E(L_T)}{\partial \eta} = \frac{1}{\theta} \frac{\partial}{\partial \eta/\theta} E(L_T)$$

$$= \frac{1}{\theta} \frac{\partial}{\partial \eta/\theta} \left\{ \frac{A_0 \Delta_0^2}{\theta^2} \left[\frac{\mathrm{e}^{-\eta/\theta}}{\eta/\theta} - \Gamma(0, \eta/\theta) \right] + (1 - \mathrm{e}^{-\eta/\theta}) C_s + C_i \right\}$$

$$= \frac{1}{\theta} \left\{ \frac{A_0 \Delta_0^2}{\theta^2} \left[\frac{-\mathrm{e}^{-\eta/\theta}}{(\eta/\theta)^2} - \frac{\mathrm{e}^{-\eta/\theta}}{\eta/\theta} + \frac{\mathrm{e}^{-\eta/\theta}}{\eta/\theta} \right] + C_s \mathrm{e}^{-\eta/\theta} \right\}$$

$$= \frac{\mathrm{e}^{-\eta/\theta}}{\theta} \left(C_s - \frac{A_0 \Delta_0^2}{\eta^2} \right)$$

Solving $\partial E(L_T)/\partial \eta = 0$ for η in the domain $0 < \eta < \infty$ gives

$$\eta = \sqrt{A_0 \Delta_0^2 / C_s}$$

Since

$$\left. \frac{\partial^2 E(L_T)}{\partial \eta^2} \right|_{\eta = \sqrt{A_0 \Delta_0^2/C_s}} = \left. \frac{-e^{-\eta/\theta}[C_s \eta^3 - A_0 \Delta_0^2 (\eta + 2\theta)]}{\eta^3 \theta^2} \right|_{\eta = \sqrt{A_0 \Delta_0^2/C_s}}$$

$$= \frac{2\theta C_s \exp(-\sqrt{A_0 \Delta_0^2/C_s}/\theta)}{A_0 \Delta_0^2 / C_s}$$

$$> 0 \text{ (because } \theta, C_s, A_0 \text{ and } \Delta_0 \text{ are all positive)}$$

this turning point is a minimum. So the value of η that minimizes the exact expected total loss per item is

$$\eta_{\min} = \sqrt{A_0 \Delta_0^2 / C_s}$$

5 Numerical example

Using the example in Chen (1999), we wish to decide the lower specification limit for the lifetime of a product, which is clearly a 'larger the better' situation. The lifetime of the product is known to be exponentially distributed with a mean of $\theta = 2500$ hours, $A_0 = \$2$, $\Delta_0 = 1000$ hours and $C_s = \$1$.

The lower specification limit that minimizes the exact expected total loss per item is simply

$$\eta_{\min} = \sqrt{A_0 \Delta_0^2 / C_s} = \sqrt{2 \times 1000^2 / 1} = 1414.2 \text{ hours}$$

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